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Werner Fenchel

Elementary Geometry in Hyperbolic Space

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Editorial

Habent Sua Fata Libelli. This proverbial statement of Terentianus Maurus applies in a very particular sense to the present volume of the de Gruyter Studies in Mathematics.

Werner Fenchel, Professor emeritus at the University of Copenhagen, one of the pioneers in the theory of convex bodies and in duality theory for convex functions, well-known also by his contributions to global differential geometry and other fields of mathematics, died on January 24, 1988 quite unexpectedly. At that time it was clear to both the publisher and the editors that Fenchel's monograph "Elementary Geometry in Hyperbolic Space" would be accepted for the de Gruyter Studies. Unfortunately, the publisher's official letter of acceptance did not reach Werner Fenchel in time.

Fenchel's monograph appears in a period of revival of interest in hyperbolic geometry. The book contains a substantial account of the parts of the theory basic to the study of Kleinian groups. But Fenchel has a lot more to say about a subject which always belonged to his favorite fields. Consequently, the publisher and editors are convinced that Werner Fenchel's monograph will interest mathematicians working in the field of hyperbolic geometry, those needing a handy reference and others who intend to study the subject.

It was a happy coincidence that Dr. Christian Siebeneicher of the University of Bielefeld was in close scientific contact with Werner Fenchel during preparation of the manuscript. Thanks to Siebeneicher's effort, Fenchel's manuscript was brought into its final form for printing.

Werner Fenchel was born in Berlin in 1905. In Berlin he studied mathematics and physics during the years 1923–1928. He had to leave Germany in 1933, a few months after Hitler's take-over. The publisher and editors are happy that – more than half a century later – Werner Fenchel's last mathematical work will appear in his hometown Berlin.

January 1989 *Heinz Bauer*

Preface

There exist many excellent books on non-Euclidean geometry. To add another one is motivated by the fact that these books contain very little about the geometry in hyperbolic space which has found various applications. Most of what is known is to be found in very old, not easily accessible papers. Many of these have shortcomings. In general the proofs do not cover exceptional cases, often of special interest. Sometimes one refers to a passage to the limit, an exact proof of which may be tedious. There are also cases in which it is not even obvious what can be expected to be valid in the limit. Further, it had already been noted in old papers that satisfactory results in the geometry of lines, which actually covers the elementary geometry of the hyperbolic plane, can only be obtained if oriented lines are considered. The proofs applied yield however often only squares of the relations aimed at, and the determination of the correct signs is not convincing.

In the following presentation of various aspects of the elementary geometry in hyperbolic space the axiomatic point of view has been completely neglected. Everything is based on the conformal model. The use of the projective model would, of course, have made it possible to obtain some of the results as consequences of theorems of projective geometry. Many of the relations would formally be simpler, but their geometric interpretation more involved. In any case, if only one model is to be used, the conformal one seems to be preferable. The tools which are needed are very modest: apart from the elements of algebra and analysis, only few facts from elementary Euclidean geometry. What goes beyond can be found in the first chapter.

The intention to prove statements under the weakest assumptions made it frequently necessary to replace well-known simple proofs, valid only under certain restrictions, by others, or to deal with exceptional cases separately. Therefore, the reader will certainly find a considerable part of the exposition rather elaborate. If it were to serve as a text-book and not primarily as a reference, much of the content could have been stated in the form of problems. The author hopes that the references to the various sections and the index make it possible to pick out what one is interested in and to skip the rest up to a few definitions and results.

Clearly most of the statements are known, in any case under more restrictive assumptions or for the plane. The notes to the various chapters contain some historical remarks and references. To trace everything back to its origin would be an impossible task. Therefore only the sources of ideas which have been used and which seem not to be well known have been quoted. There are, of course, also

some references to publications in which the subject in question is developed further.

For efficient help in many respects the author offers cordial thanks to Christian Siebeneicher, Bielefeld. Sincere thanks are also due to Mrs. Obershelp for typing the manuscript.

August 1986 *Werner Fenchel*

Contents

The reader should take notice of the following:

In Chapters I, II, III all terms denoting geometrical notions are to be understood in the Euclidean sense. In Chapter III those denoting notions of hyperbolic geometry are provided with the prefix *h*. In the following chapters terms denoting geometrical notions are to be understood in the sense of hyperbolic geometry. Those denoting Euclidean notions are provided with prefix *e*.

The values of square roots of positive numbers are always assumed to be positive.

I. Preliminaries

I.1. Quaternions

Let $\mathbb{C}$ denote the field of complex numbers. As customary, the imaginary unit is denoted by i and the complex conjugate of $a \in \mathbb{C}$ by $\bar{a}$. Also $\operatorname{Re} a := \frac{1}{2}(a + \bar{a})$, $\operatorname{Im} a := \frac{1}{2} i(\bar{a} - a)$.

We consider the set $\mathbb{C} \times \mathbb{C}$ of pairs $\mathrm{a} = (a, \alpha)$ of complex numbers. Addition of pairs is defined in the usual way:

$$\mathrm{a} + \mathrm{b} = (a, \alpha) + (b, \beta) = (a + b, \alpha + \beta),$$

so $(\mathbb{C} \times \mathbb{C}, +)$ is an abelian group. Multiplication is defined by

$$\mathrm{ab} = (a, \alpha)(b, \beta) = (ab - \alpha\bar{\beta}, a\beta + \alpha\bar{b}).$$

The distributive law is obviously satisfied, and it is easily verified that the multiplication is assciative. Further, it is seen that $(1, 0)$ acts as unity and that the pairs $(a, 0)$ form a field isomorphic with $\mathbb{C}$. Therefore we may write a instead of $(a, 0)$. Since $(0, \alpha) = (\alpha, 0)(0, 1) = \alpha(0, 1)$,

$$(a, \alpha) = a + \alpha(0, 1)$$

or, with the notation $(0, 1) = j$,

$$(a, \alpha) = a + \alpha j.$$

Computations with these *quaternions* can now be performed according to the usual rules together with the special cases

$$j^2 = -1, \quad ja = \bar{a}j \quad \text{for} \quad a \in \mathbb{C}$$

of the definition of the multiplication.

The set $\mathbb{C} \times \mathbb{C}$ provided with this ring structure will be denoted by $\mathbb{H}$.

Defining the *conjugate* of a quaternion $\mathrm{a} = a + \alpha j$ by

$$\bar{\mathrm{a}} = \bar{a} - \alpha j$$

one has

$$\mathrm{a}\bar{\mathrm{a}} = \bar{\mathrm{a}}\mathrm{a} = a\bar{a} + \alpha\bar{\alpha} \geqq 0$$

with equality if and only if $\mathrm{a} = 0$. This shows that every quaternion

$\mathrm{a} = a + \alpha j \neq 0$ has a reciprocal

$$\mathrm{a}^{-1} = \bar{\mathrm{a}}(\mathrm{a}\bar{\mathrm{a}})^{-1} = (\bar{a} - \alpha j)(a\bar{a} + \alpha\bar{\alpha})^{-1}.$$

Hence $\mathbb{H}$ is a division ring.

We notice further the following simple properties of $\mathbb{H}$: A quaternion $\mathrm{a} = a + \alpha j$ commutes with every quaternion if and only if it is real, that is, $a \in \mathbb{R}$ and $\alpha = 0$. For two quaternions a and b we have

$$\overline{\mathrm{ab}} = \bar{\mathrm{b}}\bar{\mathrm{a}}$$

which, because of $\mathrm{b}\bar{\mathrm{b}} \in \mathbb{R}$, implies

$$\mathrm{ab}\overline{\mathrm{ab}} = \mathrm{a}\bar{\mathrm{a}}\,\mathrm{b}\bar{\mathrm{b}}.$$

$\mathbb{H}$ may be considered as a real vector space of dimension 4 with the basis $(1, i, j, ij)$. The Euclidean *norm* of $\mathrm{x} \in \mathbb{H}$ may be written

$$|\mathrm{x}| = (\mathrm{x}\bar{\mathrm{x}})^{\frac{1}{2}}$$

and the *inner product* of x and y

$$\langle \mathrm{x}, \mathrm{y} \rangle = \tfrac{1}{2}((\mathrm{x} + \mathrm{y})(\bar{\mathrm{x}} + \bar{\mathrm{y}}) - \mathrm{x}\bar{\mathrm{x}} - \mathrm{y}\bar{\mathrm{y}}) = \tfrac{1}{2}(\mathrm{x}\bar{\mathrm{y}} + \mathrm{y}\bar{\mathrm{x}}).$$

The norm satisfies

$$|\mathrm{xy}| = |\mathrm{x}|\,|\mathrm{y}|.$$

The basis $(1, i, j, ij)$ is orthonormal. It will be assumed that $\mathbb{H}$ is provided with the orientation determined by it.

For applications later on it is convenient to extend $\mathbb{H}$ by adjoining to it one element, denoted ∞. Let

$$\mathbb{H}_\infty = \mathbb{H} \cup \{\infty\}.$$

For computation with ∞ the following rules are adopted:

$$\begin{aligned} &\infty^{-1} = 0,\ 0^{-1} = \infty,\ -\infty = \infty,\ \overline{\infty} = \infty, \\ &\mathrm{x} + \infty = \infty + \mathrm{x} = \infty \qquad \text{for} \quad \mathrm{x} \in \mathbb{H}, \\ &\mathrm{x}\infty = \infty\mathrm{x} = \infty \qquad\quad\ \ \text{for} \quad \mathrm{x} \in \mathbb{H}_\infty \setminus \{0\}. \end{aligned}$$

$\mathbb{H}_\infty$ may be provided with the topology the open sets of which are the open subsets of $\mathbb{H}$ and the complements with respect to $\mathbb{H}_\infty$ of the compact subsets of $\mathbb{H}$. Then $\mathbb{H}_\infty$ is a compact Hausdorff space homeomorphic to a 4-sphere and thus orientable. Obviously, $\mathrm{x} \mapsto \mathrm{x} + \mathrm{a}$ for $\mathrm{a} \in \mathbb{H}$, $\mathrm{x} \mapsto \mathrm{bx}$ and $\mathrm{x} \mapsto \mathrm{xb}$ for $\mathrm{b} \in \mathbb{H} \setminus \{0\}$, $\mathrm{x} \mapsto \mathrm{x}^{-1}$, $\mathrm{x} \mapsto \bar{\mathrm{x}}$ are topological mappings of $\mathbb{H}_\infty$ onto itself.

The subspace of $\mathbb{H}$ spanned by 1 and i is the complex plane $\mathbb{C}$ and

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$$

the extended complex plane in the usual sense. The 3-dimensional subspace of $\mathbb{H}$ generated by 1, i and j, that is,

$$\begin{aligned}\mathbb{J} &= \{\mathrm{x} + \xi j \in \mathbb{H} \mid \xi \in \mathbb{R}\}\\ &= \{\mathrm{x} \in \mathbb{H} \mid ij\mathrm{x} = \bar{\mathrm{x}}ij\}\end{aligned}$$

and its extension

$$\mathbb{J}_\infty = \mathbb{J} \cup \{\infty\},$$

which is homeomorphic to a 3-sphere, will play an essential role in the sequel. For the sake of brevity the points of $\mathbb{J}$ will be called *$\mathbb{J}$-quaternions*. The $\mathbb{J}$-quaternions $x + \xi j$ with $x \in \mathbb{R}$ form a field isomorphic with $\mathbb{C}$.

We shall write $\mathbb{R}_\infty$ for the extended real line $\mathbb{R} \cup \{\infty\}$.

I.2. The hyperbolic functions

Consider the additive group

$$\mathbb{A} = \mathbb{C}/(2\pi i\mathbb{Z}) = \mathbb{R} \oplus i\mathbb{R}/(2\pi\mathbb{Z})$$

of the complex numbers modulo $2\pi i$. Provided with the ordinaty topology it is homeomorphic to an open cylinder and will be called the *complex cylinder*. It may be compactified by adjoining two points, $+\infty$ and $-\infty$, and defining a basis for the open sets to consist of the open subsets of $\mathbb{A}$ and the sets

$$\{\xi \in \mathbb{A} \mid \operatorname{Re} \xi > \varrho\} \cup \{+\infty\}, \quad \{\xi \in \mathbb{A} \mid \operatorname{Re} \xi > \varrho\} \cup \{-\infty\}$$

for all $\varrho \in \mathbb{R}$. The *extended complex cylinder*

$$\mathbb{A}_\infty = \mathbb{A} \cup \{+\infty, -\infty\}$$

is homeomorphic to a 2-sphere. To avoid confusion with the point ∞ of $\mathbb{C}_\infty$, the points $+\infty$ and $-\infty$ of $\mathbb{A}_\infty$ will always be written with their signs.

In $\mathbb{A}$ all the usual rules for addition, subtraction and multiplication by integers are valid. Division by integers is multivalued. In the following only division by 2 is needed. For each $\xi \in \mathbb{A}$ there are two elements ζ differing by πi of $\mathbb{A}$ such that $2\zeta = \xi$. If ξ is not specified, we write $\frac{1}{2}\xi$ for one of them, choosen arbitrarily but once for all in the investigation in question. However, if $\xi \in \mathbb{A}$ is determined by a specified complex number, $\frac{1}{2}\xi$ shall denote the element of $\mathbb{A}$ determined by $\frac{1}{2}$ of this number (for example, $\xi = \pi i, \frac{1}{2}\xi = \frac{1}{2}\pi i$). Clearly, it makes sense to talk of the conjugate $\bar{\xi}$ of an element ξ of $\mathbb{A}$.

For computation with $+\infty$ and $-\infty$ the obvious conventions are adopted:

$$\begin{aligned}
+\infty+\xi &= +\infty && \text{for } \xi\in\mathbb{A}_\infty\backslash\{-\infty\},\\
-\infty+\xi &= -\infty && \text{for } \xi\in\mathbb{A}\backslash\{+\infty\},\\
-(+\infty) &= -\infty, && -(-\infty)=+\infty,\\
\tfrac{1}{2}(+\infty) &= +\infty, && \tfrac{1}{2}(-\infty)=-\infty,\\
\overline{+\infty} &= +\infty, && \overline{-\infty}=-\infty.
\end{aligned}$$

The complex cylinder $\mathbb{A}$ may be considered as the domain of the exponential function, and if we define

$$\exp(-\infty)=0, \qquad \exp(+\infty)=\infty,$$

then

$$\exp\colon \mathbb{A}_\infty \to \mathbb{C}_\infty$$

is a homeomorphism with inverse

$$\log\colon \mathbb{C}_\infty \to \mathbb{A}_\infty.$$

The hyperbolic functions

$$\begin{aligned}
\sinh\xi &= \tfrac{1}{2}(e^\xi - e^{-\xi}), & \cosh\xi &= \tfrac{1}{2}(e^\xi + e^{-\xi}),\\
\tanh\xi &= (e^{2\xi}-1)/(e^{2\xi}+1), & \coth\xi &= (e^{2\xi}+1)/(e^{2\xi}-1)
\end{aligned}$$

with

$$\begin{aligned}
\sinh(-\infty) &= \sinh(+\infty)=\infty, & \cosh(-\infty) &= \cosh(+\infty)=\infty,\\
\tanh(-\infty) &= \coth(-\infty)=-1, & \tanh(+\infty) &= \coth(+\infty)=1,\\
\tanh\tfrac{1}{2}\pi i &= \tanh(-\tfrac{1}{2}\pi i)=\infty, & \coth 0 &= \coth\pi i=\infty
\end{aligned}$$

map $\mathbb{A}_\infty$ onto $\mathbb{C}_\infty$. Each of them determines a 2-sheeted covering of $\mathbb{C}_\infty$ with 2 branch points, namely i and $-i$ for sinh and 1 and -1 for the three other functions. Thus the inverse functions are double-valued. We shall only use them for real arguments and with their "principal values" defined as follows: Let $x\in\mathbb{C}_\infty$ be restricted to $\mathbb{R}$ and $\xi\in\mathbb{A}$ to $\mathbb{R}\cup\{-\infty,+\infty\}$. Then

$\xi=\operatorname{Arsinh}x$ for $x\in\mathbb{R}$ means $\xi\in\mathbb{R}$ and $\sinh\xi=x$.

$\xi=\operatorname{Arcosh}x$ for $x\geqq 1$ means $0\leqq\xi<+\infty$ and $\cosh\xi=x$.

$\xi=\operatorname{Arthan}x$ for $-1\leqq x\leqq 1$ means $-\infty<\xi\leqq+\infty$ and $\tanh\xi=x$.

$\xi=\operatorname{Arcoth}x$ for $x\leqq -1, x\geqq 1$ means $-\infty\leqq\xi<0, 0<\xi\leqq+\infty$ and $\coth\xi=x$.

The trigonometric functions

$$\begin{aligned}
\sin\xi &= -i\,\sinh\xi i, & \cos\xi &= \cosh\xi i,\\
\tan\xi &= -i\,\tanh\xi i, & \cos\xi &= i\coth\xi i
\end{aligned}$$

will only be used for ξ real. The principal values of the inverse function to be used are

$$\begin{aligned}
&\xi = \operatorname{Arcsin} x, && -1 \leqq x \leqq 1, && -\frac{\pi}{2} \leqq \xi \leqq \frac{\pi}{2},\\
&\xi = \operatorname{Arccos} x, && -1 \leqq x \leqq 1, && \pi \geqq \xi \geqq 0,\\
&\xi = \operatorname{Arctan} x, && -\infty \leqq x \leqq +\infty, && -\frac{\pi}{2} \leqq \xi \leqq \frac{\pi}{2},\\
&\xi = \operatorname{Arccot} x, && -\infty \leqq x \leqq +\infty, && \pi \geqq \xi \geqq 0.
\end{aligned}$$

Below there is given a list of elementary relations for the hyperbolic functions which will be applied later on, often without reference:

For $\xi \in \mathbb{A}_\infty$:

$$\begin{aligned}
\sinh(-\xi) &= -\sinh\xi, & \cosh(-\xi) &= \cos\xi,\\
\tanh(-\xi) &= -\tanh\xi, & \coth(-\xi) &= -\coth\xi,\\
\sinh(\xi+\pi i) &= -\sinh\xi, & \cosh(\xi+\pi i) &= -\cosh\xi,\\
\tanh(\xi+\pi i) &= \tanh\xi, & \coth(\xi+\pi i) &= \coth\xi,\\
\sinh(\xi+\tfrac{1}{2}\pi i) &= i\cosh\xi, & \cosh(\xi+\tfrac{1}{2}\pi i) &= i\sinh\xi,\\
\tanh(\xi+\tfrac{1}{2}\pi i) &= \coth\xi, & \coth(\xi+\tfrac{1}{2}\pi i) &= \tanh\xi.
\end{aligned}$$

For $\xi \in \mathbb{A}$:

$$\cosh^2\xi - \sinh^2\xi = 1,$$

and for $\xi \in \mathbb{A}_\infty$:

$$\begin{aligned}
&\tanh\xi\,\coth\xi = 1,\\
&1-\tanh^2\xi = 1/\cosh^2\xi, \quad \coth^2\xi - 1 = 1/\sinh^2\xi.
\end{aligned}$$

For $\xi, \eta \in \mathbb{A}$:

$$\begin{aligned}
\sinh(\xi+\eta) &= \sinh\xi\,\cosh\eta + \cosh\xi\,\sinh\eta,\\
\sinh(\xi-\eta) &= \sinh\xi\,\cosh\eta - \cosh\xi\,\sinh\eta,\\
\cosh(\xi+\eta) &= \cosh\xi\,\cosh\eta + \sinh\xi\,\sinh\xi,\\
\cosh(\xi-\eta) &= \cosh\xi\,\cosh\eta - \sinh\xi\,\sinh\xi
\end{aligned}$$

and for $\xi, \eta \in \mathbb{A}_\infty$ whenever the expressions on the left-hand sides make sense:

$$\begin{aligned}
\tanh(\xi+\eta) &= (\tanh\xi + \tanh\eta)/(1+\tanh\xi\,\tanh\eta),\\
\tanh(\xi-\eta) &= (\tanh\xi - \tanh\eta)/(1-\tanh\xi\,\tanh\eta),\\
\coth(\xi+\eta) &= (\coth\xi\,\coth\eta + 1)/(\coth\eta + \coth\xi),\\
\coth(\xi-\eta) &= (\coth\xi\,\coth\eta - 1)/(\coth\eta - \coth\xi).
\end{aligned}$$

Immediate consequences of these addition formulas are:

$$\begin{aligned}
\sinh 2\xi &= 2\sinh\xi\cosh\xi = 2\tanh\xi/(1-\tanh^2\xi),\\
\cosh 2\xi &= \cosh^2\xi+\sinh^2\xi = 2\cosh^2\xi-1 = 2\sinh^2\xi+1\\
&= (1+\tanh^2\xi)/(1-\tanh^2\xi),\\
\sinh 3\xi &= 4\sinh^3\xi+3\sinh\xi, \quad \cosh 3\xi = 4\cosh^3\xi-3\cosh\xi,\\
\tanh 2\xi &= 2\tanh\xi/(1+\tanh^2\xi), \quad \coth 2\xi = (\coth^2\xi+1)/(2\coth\xi),\\
\tanh\xi &= \sinh 2\xi/(\cosh 2\xi+1) = (\cosh 2\xi-1)/\sinh 2\xi.
\end{aligned}$$

These relations are valid for $\xi \in \mathbb{A}$, some of them clearly for $\xi \in \mathbb{A}_\infty$.

For $\xi, \eta \in \mathbb{A}$ and any choice of $\frac{1}{2}\xi$ and $\frac{1}{2}\eta$:

$$\begin{aligned}
\sinh\xi + \sinh\eta &= 2\sinh\tfrac{1}{2}(\xi+\eta)\cosh\tfrac{1}{2}(\xi-\eta),\\
\sinh\xi - \sinh\eta &= 2\cosh\tfrac{1}{2}(\xi+\eta)\sinh\tfrac{1}{2}(\xi-\eta),\\
\cosh\xi + \cosh\eta &= 2\cosh\tfrac{1}{2}(\xi+\eta)\cosh\tfrac{1}{2}(\xi-\eta),\\
\cosh\xi - \cosh\eta &= 2\sinh\tfrac{1}{2}(\xi+\eta)\sinh\tfrac{1}{2}(\xi-\eta).
\end{aligned}$$

For $\xi, \eta, \zeta \in \mathbb{A}$ and any choice of $\sigma = \frac{1}{2}(\xi+\eta+\zeta)$:

$$\begin{aligned}
&4\sinh\sigma\sinh(\sigma-\xi)\sinh(\sigma-\eta)\sinh(\sigma-\zeta)\\
&\quad = 1 + 2\cosh\xi\cosh\eta\cosh\zeta - \cosh^2\xi - \cosh^2\eta - \cosh^2\zeta \qquad (1)\\
&\quad = \begin{vmatrix} 1 & \cosh\xi & \cosh\eta \\ \cosh\xi & 1 & \cosh\zeta \\ \cosh\eta & \cosh\zeta & 1 \end{vmatrix}.
\end{aligned}$$

Observing that the first expression equals

$$\begin{aligned}
&(\cosh(\eta+\zeta)-\cosh\xi)(\cosh\xi-\cosh(\zeta-\eta))\\
&\quad = \sinh^2\eta\sinh^2\zeta - (\cosh\eta\cosh\zeta-\cosh\xi)^2,
\end{aligned}$$

the relation is easily verified. It shows that the determinant vanishes if and only if

$$\xi \pm \eta \pm \zeta = 0$$

for some combination of the signs.

Another expression for the determinant above will be used later on. For convenience we replace ξ, η, ζ by $2\xi, 2\eta, 2\zeta$ and write for the sake of brevity

$$p = \sinh\xi, \quad q = \sinh\eta, \quad r = \sin\zeta.$$

Then we have

$$\begin{aligned}
&1 + 2\cosh 2\xi\cosh 2\eta\cosh 2\zeta - \cosh^2 2\xi - \cosh^2 2\eta - \cosh^2 2\zeta\\
&\quad = 1 + 2(2p^2+1)(2q^2+1)(2r^2+1) - (2p^2+1)^2 - (2q^2+1)^2\\
&\qquad\qquad - (2r^2+1)^2\\
&\quad = 16p^2q^2r^2 + 8p^2q^2 + 8q^2r^2 + 8p^2r^2 - 4p^4 - 4q^4 - 4r^4.
\end{aligned}$$

On the other hand

$$\begin{aligned}&(p+q+r)(-p+q+r)(p-q+r)(p+q-r)\\&\quad=((q+r)^2-p^2)(p^2-(q-r)^2)=4q^2r^2-(p^2-q^2-r^2)^2\\&\quad=2p^2q^2+2q^2r^2+2r^2p^2-p^4-q^4-r^4,\end{aligned}$$

hence

$$\tag{2}\begin{vmatrix}1 & \cosh 2\xi & \cosh 2\eta\\ \cosh 2\xi & 1 & \cosh 2\zeta\\ \cosh 2\eta & \cosh 2\zeta & 1\end{vmatrix}=$$

$$\begin{aligned}&=16\sinh^2\xi\,\sinh^2\eta\,\sinh^2\zeta\\&+4(\sinh\xi+\sinh\eta+\sinh\zeta)(-\sinh\xi+\sinh\eta+\sinh\zeta)\\&\cdot(\sinh\xi-\sinh\eta+\sinh\zeta)(\sinh\xi+\sinh\eta-\sinh\zeta).\end{aligned}$$

The following relation is easily verified for $\xi, \eta, \zeta \in \mathbb{A}$:

$$\tag{3}\begin{aligned}&4\cosh\xi\,\cosh\eta\,\cosh\zeta\\&\quad=\cosh(\xi+\eta+\zeta)+\cosh(-\xi+\eta+\zeta)+\cosh(\xi-\eta+\zeta)\\&\qquad+\cosh(\xi+\eta-\zeta).\end{aligned}$$

Replacing ξ, η, ζ by $\xi+\frac{\pi}{2}i$, $\eta+\frac{\pi}{2}i$, $\zeta+\frac{\pi}{2}i$, respectively, one obtains

$$\tag{4}\begin{aligned}&4\sinh\xi\,\sinh\eta\,\sinh\zeta\\&\quad=\sinh(\zeta+\eta+\zeta)-\sinh(-\xi+\eta+\zeta)-\sinh(\xi-\eta+\zeta)\\&\qquad-\sinh(\xi+\eta-\zeta),\end{aligned}$$

in particular for $\xi+\eta+\zeta=0$,

$$4\sinh\xi\,\sinh\eta\,\sinh\zeta=\sinh 2\xi+\sinh 2\eta+\sinh 2\zeta.$$

Furthermore,

$$\tanh\xi+\tanh\eta+\tanh\zeta+\tanh\xi\,\tanh\eta\,\tanh\zeta=0,$$

if and only if $\xi+\eta+\zeta=0$ or πi.

Let $a, b, c \in \mathbb{C}$, $(a,b) \neq (0,0)$ be given. Consider the equation

$$a\cosh\xi+b\sinh\xi=c$$

with the unknown $\xi \in \mathbb{A}$.

If $a^2 \neq b^2$, there exists $\varrho \in \mathbb{A}$ such that

$$\cosh\varrho=a(a^2-b^2)^{-\frac{1}{2}},\ \sinh\varrho=b(a^2-b^2)^{-\frac{1}{2}}$$

for a chosen value of the square root. The equation may thus be written

$$\cosh(\xi+\varrho)=c(a^2-b^2)^{-\frac{1}{2}}.$$

It has two solutions in $\mathbb{A}$ which coincide if and only if $a^2 - b^2 = c^2$.
If $a = b$ or $a = -b$, the equation takes the form

$$e^{\xi} = c/a \quad \text{or} \quad e^{-\xi} = c/a .$$

It has one solution $\xi \in \mathbb{A}$ provided $c \neq 0$ and the solution $-\infty$ or $+\infty$, respectively, if $c = 0$.

I.3. Trace relations

We consider the algebra of 2×2 matrices with complex elements. The unit matrix will be denoted by $\mathbf{1}$ and the zero matrix by $\mathbf{0}$. We shall write $\det \mathbf{a}$ and $\operatorname{tr} \mathbf{a}$ for the determinant and the trace of $\mathbf{a}$, respectively. Further, we shall write $\operatorname{tr} \mathbf{a}^2$ for $\operatorname{tr}(\mathbf{a}^2)$ and $\operatorname{tr}^2 \mathbf{a}$ for $(\operatorname{tr} \mathbf{a})^2$, similarly for det.

With a matrix

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we associate its *adjugate*

$$\mathbf{a}^{\sim} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

It satisfies

$$\begin{aligned} &\operatorname{tr} \mathbf{a}^{\sim} = \operatorname{tr} \mathbf{a}, && \det \mathbf{a}^{\sim} = \det \mathbf{a}, \\ &\mathbf{a} + \mathbf{a}^{\sim} = \mathbf{1} \operatorname{tr} \mathbf{a}, && \mathbf{a}\mathbf{a}^{\sim} = \mathbf{a}^{\sim}\mathbf{a} = \mathbf{1} \det \mathbf{a}, \end{aligned} \tag{1}$$

in particular

$$\mathbf{a}^{\sim} = \mathbf{a}^{-1} \det \mathbf{a} \quad \text{if} \quad \det \mathbf{a} \neq 0 .$$

It is easily verified that $^{\sim}$ determines an involutary anti-automorphism of the matrix algebra:

$$(\mathbf{a}^{\sim})^{\sim} = \mathbf{a}, \quad (\mathbf{a} + \mathbf{b})^{\sim} = \mathbf{a}^{\sim} + \mathbf{b}^{\sim}, \quad (\mathbf{ab})^{\sim} = \mathbf{b}^{\sim}\mathbf{a}^{\sim} .$$

For the traces of matrices we have

$$\operatorname{tr}(\mathbf{ab}) = \operatorname{tr}(\mathbf{ba})$$

and, using this and (1),

$$\operatorname{tr}(\mathbf{aba}^{\sim}) = \det \mathbf{a} \operatorname{tr} \mathbf{b} .$$

Multiplying the first equation (1) by $\mathbf{b}$ and taking traces, one obtains

$$\operatorname{tr}(\mathbf{ab}) + \operatorname{tr}(\mathbf{a}^{\sim}\mathbf{b}) = \operatorname{tr}\mathbf{a}\operatorname{tr}\mathbf{b}. \tag{2}$$

This relation and its special case

$$\operatorname{tr}\mathbf{a}^2 = \operatorname{tr}^2\mathbf{a} - 2\det\mathbf{a} \tag{3}$$

will be used frequently.

Another useful relation is

$$\begin{aligned}\operatorname{tr}(\mathbf{aba}^{\sim}\mathbf{b}^{\sim}) = &-2\det\mathbf{a}\det\mathbf{b} - \operatorname{tr}\mathbf{a}\operatorname{tr}\mathbf{b}\operatorname{tr}(\mathbf{ab}) + \\ &+ \operatorname{tr}^2\mathbf{a}\det\mathbf{b} + \operatorname{tr}^2\mathbf{b}\det\mathbf{a} + \operatorname{tr}^2(\mathbf{ab}).\end{aligned} \tag{4}$$

To prove it, apply (2) repeatedly:

$$\begin{aligned}\operatorname{tr}(\mathbf{aba}^{\sim}\mathbf{b}^{\sim}) &= \operatorname{tr}((\mathbf{ba})^{\sim}\mathbf{ab}) = \operatorname{tr}^2(\mathbf{ab}) - \operatorname{tr}(\mathbf{ba}^2\mathbf{b}) \\ &= \operatorname{tr}^2(\mathbf{ab}) - \operatorname{tr}\mathbf{b}\operatorname{tr}(\mathbf{a}^2\mathbf{b}) + \operatorname{tr}(\mathbf{b}^{\sim}\mathbf{a}^2\mathbf{b}) \\ &= \operatorname{tr}^2(\mathbf{ab}) - \operatorname{tr}\mathbf{a}\operatorname{tr}\mathbf{b}\operatorname{tr}(\mathbf{ab}) + \operatorname{tr}^2\mathbf{b}\det\mathbf{a} + \operatorname{tr}\mathbf{a}^2\det\mathbf{b}.\end{aligned}$$

Application of (3) to the last term yields the statement. It may also be written

$$4\det\mathbf{a}\det\mathbf{b} - 2\operatorname{tr}(\mathbf{aba}^{\sim}\mathbf{b}^{\sim}) = \begin{vmatrix} 2\det\mathbf{a} & \operatorname{tr}\mathbf{a} & \operatorname{tr}(\mathbf{ab}) \\ \operatorname{tr}\mathbf{a} & 2 & \operatorname{tr}\mathbf{b} \\ \operatorname{tr}(\mathbf{ab}) & \operatorname{tr}\mathbf{b} & 2\det\mathbf{b}\end{vmatrix}. \tag{5}$$

If, in particular, $\det\mathbf{a} = \det\mathbf{b} = \det\mathbf{c} = 1$ and $\mathbf{cba} = \mathbf{1}$, we then have

$$4 - 2\operatorname{tr}(\mathbf{abc}) = \begin{vmatrix} 2 & \operatorname{tr}\mathbf{a} & \operatorname{tr}\mathbf{c} \\ \operatorname{tr}\mathbf{a} & 2 & \operatorname{tr}\mathbf{b} \\ \operatorname{tr}\mathbf{c} & \operatorname{tr}\mathbf{b} & 2\end{vmatrix}. \tag{6}$$

Observing that

$$\operatorname{tr}(\mathbf{ab}^{\sim}) = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11},$$

one easily verifies that, for $p, q \in \mathbb{C}$,

$$\begin{aligned}\det(p\mathbf{a} + q\mathbf{b}) &= p^2\det\mathbf{a} + pq\operatorname{tr}(\mathbf{ab}^{\sim}) + q^2\det\mathbf{b} \\ &= p^2\det\mathbf{a} + pq(\operatorname{tr}\mathbf{a}\operatorname{tr}\mathbf{b} - \operatorname{tr}(\mathbf{ab})) + q^2\det\mathbf{b}.\end{aligned} \tag{7}$$

Special cases to be used later on are

$$\begin{aligned}&\det(\mathbf{a} - \mathbf{a}^{\sim}) = 4\det\mathbf{a} - \operatorname{tr}^2\mathbf{a}, \\ &\det(\mathbf{ab} - \mathbf{ba}) = 2\det\mathbf{a}\det\mathbf{b} - \operatorname{tr}(\mathbf{aba}^{\sim}\mathbf{b}^{\sim}).\end{aligned}$$

Hence, the left-hand side of (5) may be replaced by $2\det(\mathbf{ab} - \mathbf{ba})$.

I.4. The fractional linear group and the cross ratio

Consider the special linear group $\mathrm{SL}(2, \mathbb{C})$ of 2×2 matrices $\mathbf{f}$ with complex elements and $\det \mathbf{f} = 1$.

It is easily verified that every such matrix $\mathbf{f} = (f_{\lambda\mu})_{\lambda, \mu = 1,2}$ may be written as a product:

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} 1 & f_{11}/f_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & f_{22}f_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_{21} & 0 \\ 0 & 1/f_{21} \end{pmatrix} \qquad \text{if } f_{21} \neq 0,$$

$$= \begin{pmatrix} 1 & f_{11}f_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_{11} & 0 \\ 0 & 1/f_{11} \end{pmatrix} \qquad \text{if } f_{21} = 0.$$

Furthermore, for $p \in \mathbb{C} \setminus \{0\}$ one has

$$\begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix} = \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1/p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Application hereof with $p = f_{21}$ and $p = f_{11}$ in the expressions for $\mathbf{f}$ above shows that $\mathrm{SL}(2, \mathbb{C})$ is generated by the elements

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \quad q \in \mathbb{C}.$$

One infers also that these matrices with $q \in \mathbb{R}$ generate $\mathrm{SL}(2, \mathbb{R})$.

We shall be interested in the following action of this group on the extended complex plane $\mathbb{C}_\infty$. To the matrix $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ we let correspond the function f defined by

$$f(x) = \frac{f_{11}x + f_{12}}{f_{21}x + f_{22}} \qquad \text{for} \quad x \in \mathbb{C}$$

$$= \frac{f_{11}}{f_{21}} \qquad \text{for} \quad x = \infty$$

with the conventions for computing with ∞ formulated in I.1. In the sequel the value for $x = \infty$ will not be written separately. If also $\mathbf{g} \in \mathrm{SL}(2, \mathbb{C})$ it is easily checked that the function corresponding to the matrix $\mathbf{gf}$ is $g \circ f$. Applied to $\mathbf{g} = \mathbf{f}^{\sim} = \mathbf{f}^{-1}$ this shows that $f^{\sim} \circ f$ is the identity. Hence $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is bijective, and $\mathbf{f} \mapsto f$ is a homomorphism. Its kernel is $\{\mathbf{1}, -\mathbf{1}\}$ because

$$\frac{f_{11}x + f_{12}}{f_{21}x + f_{22}} = x$$

for $x = 0$ and $x = \infty$ implies $f_{12} = 0$ and $f_{21} = 0$, respectively, and then for $x = 1$ that $f_{11} = f_{22}$. Thus we have:

The fractional linear transformations f form a group isomorphic with the projective special linear group

$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\mathbf{1}, -\mathbf{1}\}.$$

Clearly, a matrix $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ may be multiplied by any non-zero complex number without changing the corresponding transformation f. Occasionally it will be convenient to admit matrices with non-vanishing determinants different from 1.

For an ordered quadruple (a, b, c, d) of points of $\mathbb{C}_\infty$ no three of which coincide the *cross ratio* is defined by

$$\not{R}(a, b, c, d) = \frac{(c-a)(d-b)}{(c-b)(d-a)} \in \mathbb{C}_\infty.$$

If one or two of the points is ∞, this is to be interpreted as follows. In the expression on the right-hand side divide the numerator and the denominator by the letter or letters which are to be replaced by ∞ and apply $1/\infty = 0$. For instance one obtains

$$\not{R}(\infty, b, c, d) = \frac{d-b}{c-b}.$$

In case two of the points coincide, the value of the cross ratio is independent of the points. Indeed one has

$$\not{R}(a, b, c, d) = 0 \quad \text{if and only if} \quad a = c \quad \text{or/and} \quad b = d,$$

as is easily seen, also if ∞ occurs. Further,

$$\not{R}(a, b, c, d) = \infty \quad \text{if and only if} \quad a = d \quad \text{or/and} \quad b = c,$$

also if ∞ occurs. Finally

$$\not{R}(a, b, c, d) = 1 \quad \text{if and only if} \quad a = b \quad \text{or/and} \quad c = d,$$

"only if" since $\not{R}(a, b, c, d) = 1$ is equivalent to $(a-b)(c-d) = 0$ provided ∞ does not occur. If, for instance, $a = \infty$ and $c \neq d$,

$$\frac{d-b}{c-b} = 1$$

is equivalent to $b = \infty$, and similarly in the other cases.

The cross ratio is invariant under fractional linear transformations, that is, for every such transformation f one has

$$\not{R}(f(a), f(b), f(c), f(d)) = \not{R}(a, b, c, d).$$

If two of the points coincide, the statement is an immediate consequence of the discussion above. We may therefore suppose that the four points are distinct. Since the group of linear fractional transformations is generated by those corresponding to the matrices $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$, $q \in \mathbb{C}$, and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, it suffices to consider $x \mapsto x+q$ and $x \mapsto -1/x$. The invariance under $x \mapsto x+q$ is obvious, also if one of the points is ∞. If a, b, c, d are all different from 0 and ∞, the invariance under $x \mapsto -1/x$ is also clear. If, for instance, $a = \infty$ one has

$$R(0, -1/b, -1/c, -1/d) = \frac{-1/c}{-1/b+1/c} \cdot \frac{-1/d+1/b}{-1/d} = \frac{d-b}{c-b},$$

as stated. Similarly the cases where one of the points is 0 are checked.

The following rules are valid whenever no three of the points a, b, c, d coincide:

$$\begin{aligned} &R(c, d, a, b) = R(a, b, c, d), \\ &R(b, a, c, d) = 1/R(a, b, c, d), \\ &R(a, c, b, d) = 1 - R(a, b, c, d). \end{aligned}$$

The first and the second are obvious, even if ∞ occurs, and the third is easily checked.

These relations show that the cross ratio of any permutation of (a, b, c, d) depends only on $r = R(a, b, c, d)$:

$$\begin{aligned} r &= R(a,b,c,d) = R(c,d,a,b) = R(b,a,d,c) = R(d,c,b,a), \\ 1/r &= R(b,a,c,d) = R(c,d,b,a) = R(a,b,d,c) = R(d,c,a,b), \\ 1-r &= R(a,c,b,d) = R(b,d,a,c) = R(c,a,d,b) = R(d,b,c,a), \\ (r-1)/r &= R(b,c,a,d) = R(a,d,b,c) = R(c,b,d,a) = R(d,a,c,b), \\ r/(r-1) &= R(c,b,a,d) = R(a,d,c,b) = R(b,c,d,a) = R(d,a,b,c), \\ 1/(1-r) &= R(c,a,b,d) = R(b,d,c,a) = R(a,c,d,b) = R(d,b,a,c). \end{aligned}$$

Let $a, b, c, d, e \in \mathbb{C}_\infty$ such that no three of a, b, c, e coincide and $d \neq a$, $d \neq b$. Then

$$R(a, b, c, d)\, R(a, b, d, e) = R(a, b, c, e).$$

Under the assumptions all three cross ratios are defined, and the lefthand side does not take the form $0 \cdot \infty$ or $\infty \cdot 0$. If the five points are distinct, the relation is an immediate consequence of the definition, also if ∞ occurs. If there are coincides, it is easily checked by means of the results above.

If a, b, c are distinct

$$f(x) = R(a, b, c, x) = \frac{(c-a)(x-b)}{(c-b)(x-a)}$$

is a fractional linear transformation with determinant

$$(c-a)(c-b)(b-a) \neq 0 .$$

This implies: For $y \in \mathbb{C}_\infty$ given, there is a unique $x \in \mathbb{C}_\infty$ such that

$$\cancel{R}(a, b, c, x) = y .$$

Further we have $f(a) = \infty, f(b) = 0, f(c) = 1$. There is no other fractional linear transformation g satisfying this, for $g(a) = \infty$, $g(b) = 0$, $g(c) = 1$ imply

$$f(x) = \cancel{R}(a, b, c, x) = \cancel{R}(\infty, 0, 1, g(x)) = g(x) .$$

From these facts we infer:

Given three distinct points a_1, b_1, c_1 and three distinct points a_2, b_2, c_2, there is a unique fractional linear transformation f which maps a_1 onto a_2, b_1 onto b_2 and c_1 onto c_2. Implicitly it is given by

$$\cancel{R}(a_1, b_1, c_1, x) = \cancel{R}(a_2, b_2, c_2, f(x)) .$$

A non-identical fractional linear transformation

$$y = (f_{11}x + f_{12})/(f_{21}x + f_{22})$$

determined by the matrix $\mathbf{f}$ with $\det \mathbf{f} = 1$ has two distinct or coinciding fixed points, namely the roots of the equation

$$f_{21}x^2 - (f_{11} - f_{22})x - f_{12} = 0 ,$$

with the convention that ∞ is a root if $f_{21} = 0$. Assume that the roots u and v are distinct, thus, that

$$(f_{11} - f_{22})^2 + 4f_{12}f_{21} = \mathrm{tr}^2\,\mathbf{f} - 4 \neq 0 .$$

Then for any x and y, both different from u and v, we have

$$\cancel{R}(v, u, x, y) = \cancel{R}(v, u, f(x), f(y)) .$$

Multiplying both sides by $\cancel{R}(v, u, y, f(x))$ we obtain

$$\cancel{R}(v, u, x, f(x)) = \cancel{R}(v, u, y, f(y)) .$$

Hence

$$\cancel{R}(v, u, x, f(x)) = m$$

is independent of x. The number m, which is different from ∞, 0, and 1, is called a *multiplier* of f. It depends on the order (u, v) in which the fixed points are taken; if this order is reversed, m has to be replaced by $1/m$, so the transformation determines $m + 1/m$.

Conjugate elements of the group have the same multipliers. To see this, observe that for any element p of the group, $p \circ f \circ p^{-1}$ has the fixed points $p(u)$ and $p(v)$. Therefore

$$m = \not{R}(v, u, p^{-1}(x), f \circ p^{-1}(x)) = \not{R}(p(v), p(u), x, p \circ f \circ p^{-1}(x))$$

yields the statement.

Conversely, any two fractional linear transformations with the same multipliers are conjugate. This follows from the fact that if f has multiplier m, it is conjugate to $y = mx$. In fact, let p be a transformation such that $p(u) = 0$ and $p(v) = \infty$. Then $\not{R}(v, u, x, f(x)) = m$ implies

$$m = \not{R}(v, u, p^{-1}(x), f \circ p^{-1}(x)) = \not{R}(\infty, 0, x, p \circ f \circ p^{-1}(x)) = p \circ f \circ p^{-1}(x)/x.$$

The multiplier of f is determined by the trace of a matrix $\mathbf{f}$ with $\det \mathbf{f} = 1$ determining f. To show this we use that conjugate matrices have the same trace. Indeed,

$$\det(\mathbf{f} - \mathbf{1}) = 2 - \operatorname{tr} \mathbf{f}$$

and

$$\det(\mathbf{p}\mathbf{f}\mathbf{p}^{-1} - \mathbf{1}) = \det(\mathbf{f} - \mathbf{1})$$

for any $\mathbf{p} \in \mathrm{SL}(2, \mathbb{C})$. Now $y = mx$ is determined by the matrix $\begin{pmatrix} m^{1/2} & 0 \\ 0 & m^{-1/2} \end{pmatrix}$ with any choice of the square root. Hence

$$m + 1/m = \operatorname{tr} \mathbf{f}^2 = \operatorname{tr}^2 \mathbf{f} - 2.$$

The transformations f with coinciding fixed points and different from the identity are all conjugate to $y = x + 1$ and hence, mutually conjugate. Let u be the fixed point of f and p any transformation with $p(u) = \infty$. Then $p \circ f \circ p^{-1}$ leaves ∞ fixed and is therefore of the form $y' = ax' + b$. Since there is no other fixed point, we must have $a = 1$ and $b \neq 0$. Put $g(x) = x + b$ and $q(x) = x/b$. Then $q \circ g \circ q^{-1}(x) = (bx + b)/b = x + 1$. This proves the statement.

A circle in $\mathbb{C}_\infty$ is by definition an ordinary circle in $\mathbb{C}$ or the union of a straight line in $\mathbb{C}$ and the point ∞. A circle in this sense has an equation of the form

$$\alpha x \bar{x} - \bar{b} x - b \bar{x} + \gamma = 0,$$

where $\alpha, \gamma \in \mathbb{R}$, $b \in \mathbb{C}$, $b\bar{b} - \alpha\gamma > 0$, with the convention that it is satisfied by ∞ if $\alpha = 0$. If $\alpha \neq 0$, it is an equation of the circle with centre b/α and radius $(b\bar{b} - \alpha\gamma)^{1/2}/|\alpha|$. If $\alpha = 0$, it is an equation of the line normal to b and passing through the point $\frac{1}{2}\gamma b/b\bar{b}$.

Every fractional linear transformation maps circles onto circles.

To prove this it is sufficient to consider the transformations $x \mapsto y = x + q$, $q \in \mathbb{C}$, and $x \mapsto y = -1/x$. Since the first named are translations of the complex plane, the statement is obvious. Substitution of $x = -1/y$ in the equation above

and multiplication by $y\bar{y}$ yield

$$\alpha - \bar{b}\bar{y} - by + \gamma y\bar{y} = 0$$

and thus the statement.

Four points a, b, c, d no three of which coincide lie in a circle in $\mathbb{C}_\infty$ if and only if

$$\not{R}(a, b, c, d) \in \mathbb{R}_\infty .$$

If the points are distinct and lie on a circle, the pairs (a, b) and (c, d) separate each other or do not separate each other on that circle according as $\not{R}(a, b, c, d) < 0$ or $\not{R}(a, b, c, d) > 0$.

If there are coincidences of the points, the only possible values of the cross ratio are $0, 1, \infty$, and the statement is therefore trivially true. If the points are distinct, application of the fractional linear transformation $x \mapsto \not{R}(a, b, c, x)$ shows that it is sufficient to consider the case $a = \infty$, $b = 0$, $c = 1$. Since $\not{R}(\infty, 0, 1, d) = d$, the first statement is now obvious. This holds also of the second, because the fractional linear transformations are homeomorphisms for the topology of $\mathbb{C}_\infty$ introduced in I.1 and therefore preserve separation of point pairs on a circle.

Two point pairs (a, b) and (c, d) are said to be *harmonic* if the four points are distinct and

$$\not{R}(a, b, c, d) = -1$$

or if three of the points coincide and are different from the fourth.

The conditions are seen to be independent of the order in which the pairs are taken and also of the order of the points in each of the pairs.

If the four points are distinct and different from ∞, the condition is equivalent to

$$(c - a)(d - b) + (d - a)(c - b) = 0,$$

which may be written

$$2(ab + cd) - (a + b)(c + d) = 0,$$

and if one of them, say a, is ∞, it is equivalent to

$$2b = c + d.$$

In both cases these conditions are satisfied if three of the points coincide.

It will be convenient to write

$$\not{R}(a, b, c, d) = -1$$

also if three of the points coincide but are different from the fourth.

Given two non-ordered pairs of points (a, b) and (c, d) which do not consist of

the same points and such that no three of the four points coincide, there exists one and only one non-ordered pair of points (x, y) which is harmonic with both (a, b) and (c, d).

If $a = b$ and $c = d$, the pair (a, c) and no other satisfies the requirement. Assume that at least one of the pairs, say (a, b), consists of distinct points. Because of the invariance of the statement under fractional linear transformations it may be assumed that $a = \infty$ and $b = 0$. The condition $\not R(a, b, x, y) = -1$ then reduces to $y = -x$, and consequently $\not R(c, d, x, y) = -1$ to $x^2 = cd$. Hence the pair required is $(-(cd)^{1/2}, (cd)^{1/2})$, which has to be interpreted as (∞, ∞) if one of c and d is ∞.

Notes to Chapter I

The contents of Sections 1, 2, and 4 are well-known and to be found in various text books. This does not hold of the content of Section 3. Trace relations for (2×2)-matrices with determinant 1 were found by Fricke [8]. For the applications in the following chapters it was necessary to generalize them to arbitrary (2×2)-matrices and to add some new ones.

II. The Möbius Group

II.1 Similarity transformations

Let $\mathbb{E}^3$ denote the Euclidean 3-space, and let there be chosen an orthonormal coordinate system with origin o and base vectors e_1, e_2, e_3. The point or vector with coordinates x_1, x_2, x_3 will be denoted by x and the inner product of two vectors x and y by $\langle x, y\rangle$. The norm of a vector x is $|x| := \langle x, x\rangle^2$. Let further $\mathbb{E}^2$ be the plane in $\mathbb{E}^3$ consisting of the points with vanishing third coordinate, that is, the plane through o spanned by e_1 and e_2.

A *similarity* of $\mathbb{E}^2$ or $\mathbb{E}^3$ is a transformation which can be written

$$x \mapsto c\mathbf{A}x + a\,,$$

where c is a positive number, $\mathbf{A}$ an orthogonal matrix, a a vector, and where x and a are interpreted as column matrices. A similarity is *direct*, that is orientation preserving, or *opposite*, that is orientation reversing, according as $\det \mathbf{A} = 1$ or $= -1$.

A *flag F in* $\mathbb{E}^2$ is by definition a configuration consisting of an ordered pair of distinct points and one of the open half-planes bounded by the line joining the points. The *basic flag* F_0 consists of the points o and $o + e_1$ and the half-plane bounded by the first coordinate axis and containing $o + e_2$.

A *flag F in* $\mathbb{E}^3$ is a configuration consisting of an ordered pair of distinct points, an open half-plane bounded by the line joining the points and one of the open half-spaces bounded by the plane containing the half-plane. The *basic flag* F_0 consists of the points o and $o + e_1$, the half-plane bounded by the first coordinate axis and containing $o + e_2$ and the half-space bounded by $\mathbb{E}^2$ and containing $o + e_3$.

From the well-known properties of the isometry groups of $\mathbb{E}^2$ and $\mathbb{E}^3$ one infers:

(1) Given two flags F_1 and F_2 in $\mathbb{E}^2$ (in $\mathbb{E}^3$), there exists one and only one similarity of $\mathbb{E}^2$ (of $\mathbb{E}^3$) which maps F_1 onto F_2.

A similarity of $\mathbb{E}^2$ maps lines onto lines and circles onto circles. A similarity of $\mathbb{E}^3$ maps planes onto planes and spheres onto spheres. Conversely:

(2) Every bijection of $\mathbb{E}^2$ which maps lines onto lines and circles onto circles is a similarity.
Every bijection of $\mathbb{E}^3$ which maps planes onto planes and spheres onto spheres is a similarity.

This is an immediate consequence of the classical theorem that a bijection of $\mathbb{E}^2$ or $\mathbb{E}^3$ mapping lines onto lines is an affine transformation. We give a simple proof not using this.

A transformation ψ of $\mathbb{E}^2$ satisfying the assumptions maps a half-plane onto a half-plane since two points belong to the same half-plane if and only if there passes a circle through them which does not intersect the line bounding the half-plane. Similarly it is seen that ψ maps the interior of a circle onto the interior of a circle. Further, a transformation ψ of $\mathbb{E}^3$ satisfying the assumptions maps a half-space onto a half-space, a half-plane onto a half-plane, and the interior of a sphere onto the interior of a sphere. Hence, in both cases a flag is mapped onto a flag.

In both cases parallel lines are mapped onto parallel lines, hence the vertices of a parallelogram onto the vertices of a parallelogram. Since a parallelogram is a rectangle if and only if the vertices lie on a circle, the vertices of a rectangle are mapped onto the vertices of a rectangle. This implies that orthogonality of lines is preserved. Since a square is characterized as a rectangle with orthogonal diagonals, the vertices of a square, its centre and the midpoints of its sides are mapped onto the corresponding points of a square.

The transformation ψ maps the basic flag F_0 onto a flag F. There is a similarity φ which does the same, so $\varphi^{-1} \circ \psi$ maps F_0 onto itself. We have to show that $\varphi^{-1} \circ \psi$ is the identity. Consider first the case of $\mathbb{E}^2$. Since the points $(0,0)$ and $(1,0)$ are fixed and each of the half-planes bounded by the first coordinate axis is mapped onto itself, the remark above about squares shows that the points $(0,1)$, $(1,1)$, $(0,-1)$, $(1,-1)$ are also fixed. One infers successively that all points whose coordinates are integers, half-integers, fractions with powers of 2 as denominators are fixed. Since any point of $\mathbb{E}^2$ is interior to an arbitrarily small circle through three fixed points, it must be fixed. In the case of $\mathbb{E}^3$ the same argument shows that $\varphi^{-1} \circ \psi$ must leave $\mathbb{E}^2$ pointwise fixed. Any point of $\mathbb{E}^3$ not in $\mathbb{E}^2$ is a vertex of a square which lies in a plane orthogonal to $\mathbb{E}^2$ and has two neighbouring vertices in $\mathbb{E}^2$. Since orthogonality is preserved and each of the half-spaces bounded by $\mathbb{E}^2$ is mapped onto itself, the vertices of the square must be fixed. Hence $\varphi^{-1} \circ \psi$ is the identity also in this case.

II.2 The extended space. Orientation. Angular measure

We adjoin one point ∞, to $\mathbb{E}^3$ and provide

$$\mathbb{E}^3_\infty := \mathbb{E}^3 \cup \{\infty\}$$

with the topology the open sets of which are the open subsets of $\mathbb{E}^3$ and the complements with respect to $\mathbb{E}^3_\infty$ of the compact subsets of $\mathbb{E}^3$. Then $\mathbb{E}^3_\infty$ is

homeomorphic to a 3-sphere and

$$\mathbb{E}^2_\infty := \mathbb{E}^2 \cup \{\infty\}$$

to a 2-sphere, and both are orientable.

By definition, a *sphere in* $\mathbb{E}^3_\infty$ is a sphere in $\mathbb{E}^3_\infty$ in the usual sense or the union of a plane in $\mathbb{E}^3_\infty$ and $\{\infty\}$. A *ball in* $\mathbb{E}^3_\infty$ is a domain bounded by a sphere, thus the interior or the exterior including ∞ of a sphere in the usual sense, or an open halfspace of $\mathbb{E}^3_\infty$.

A *circle in* $\mathbb{E}^3_\infty$ is a circle in $\mathbb{E}^3_\infty$ in the usual sense or the union of a line in $\mathbb{E}^3_\infty$ and $\{\infty\}$. A *cap in* $\mathbb{E}^3_\infty$ is a subdomain of a sphere bounded by a circle on that sphere. A cap which lies in a plane will also be called a *disk*. It is the interior or the exterior including ∞ of a circle of that plane, or an open half-plane.

A *spherical flag in* $\mathbb{E}^3_\infty$ is a configuration $F = (p, q, r, C, B)$ consisting of an ordered triplet of distinct points p, q, r, a cap C bounded by the circle through these points, and a ball B bounded by the sphere containing C.

A *circular flag in* $\mathbb{E}^2_\infty$ is a configuration $F = (p, q, r, D)$ consisting of an ordered triple of distinct points p, q, r and a disk D bounded by the circle through these points.

The spherical and circular flags whose first points p are ∞ will be identified with the flags in $\mathbb{E}^3$ and $\mathbb{E}^2$, as defined in II.1, obtained by omitting ∞. The spherical or circular flag identified in this manner with the basic flag F_0 in $\mathbb{E}^3$ or $\mathbb{E}^2$ will be called the *basic flag* in $\mathbb{E}^3_\infty$ or $\mathbb{E}^2_\infty$ and will likewise be denoted by F_0.

A spherical flag (p, q, r, C, B) determines an *orientation of* $\mathbb{E}^3_\infty$ in the following way: Let the circle through p, q, r be oriented in accordance with the cyclical order of these points. At an arbitrary point of the circle choose a tangent vector u in the positive direction, a vector v orthogonal to u and tangent to the cap C, and a vector w orthogonal to u and v pointing into the ball B. If p, q, r are collinear, thus C a half-plane and B a half-space, the point ∞ may be choosen. By a vector at ∞ we mean a directed half-line with initial point ∞. Let u, v, w denote such half-lines for a flag of this kind, and let u', v', w' be vectors for the same flag but with a finite initial point on the line through p, q, r, then the directions of u and u' agree, while those of v and v' as well as those of w and w' are opposite. In all cases the orthogonal frame (u, v, w) determines an orientation which does not depend on the choice of the initial point. The orientation will be called positive or negative according as it does or does not agree with that determined by the basic flag.

Similarly a circular flag (p, q, r, D) determines an *orientation of* $\mathbb{E}^2_\infty$ by orthogonal frames (u, v) attached to it. In the case of collinear p, q, r it has to be observed that if u, v are vectors at ∞ and u', v' vectors at a finite point for such a flag, then, seen from a finite point, the orientation of (u, v) is opposite to that of (u', v').

To any non-ordered pair of non-zero vectors in $\mathbb{E}^3_\infty$ we assign the Euclidean *angular measure* in the interval $[\![0, \pi]\!]$. The angle between two intersecting spheres is measured by means of normal vectors to them at any common point. If the

spheres are oriented in the sense that positive normal directions have been chosen, the measure is unique in $[\![0, \pi]\!]$, otherwise it may be chosen in $[\![0, \frac{1}{2}\pi]\!]$.

In the extended plane $\mathbb{E}^2_\infty$, oriented by the basic flag, the *angular measure* of an ordered pair of vectors at a point is defined in the usual manner as a real number modulo 2π. Here it has again to be taken into account that the positive sense of rotation about ∞ appears as the negative sense about a finite point. To measure the angle of an ordered pair of oriented circles which intersect one chooses one of the common points and vectors at it tangent to the circles and directed in accordance with their orientations. The measures obtained in this way at the two points of intersection are the negatives of each other.

A similarity φ of $\mathbb{E}^3$ extended to $\mathbb{E}^3_\infty$ by defining $\varphi(\infty) = \infty$ is a transformation of $\mathbb{E}^3_\infty$ preserving angular measure, obviously also at ∞. A direct similarity of $\mathbb{E}^2$ extended to $\mathbb{E}^2_\infty$ preserves angular measure while an opposite similarity reverses its sign. For the sake of brevity, transformations with these properties will be called *conformal.*

II.3 Inversion

A well-known theorem of elementary Euclidean geometry states: Let C be a circle in $\mathbb{E}^2$ and c a point in its exterior. If a line through c intersects C at x and y, then the product of the distances from c to x and y equals the square of the length of the tangent to C from c to the point of contact.

This implies: Let C_0 be a circle in $\mathbb{E}^2$ with centre c and radius ϱ. If another circle C intersects C_0 orthogonally, then any line through c which intersects C does so at two points x und y lying on the same side of c and such that

$$(1) \qquad |x-c|\,|y-c| = \varrho^2 .$$

Conversely, if a line through c *intersects a circle* C in two points with these two properties, then C intersects C_0 orthogonally. Given a sphere S_0 in $\mathbb{E}^3$ with centre c and radius ϱ, the same conditions are necessary and sufficient for a circle or a sphere to intersect S_0 orthogonally.

A slight modification hereof gives a theorem valid in the extended plane and the extended space:

Given a circle C_0 in $\mathbb{E}^2_\infty$ (a sphere S_0 in $\mathbb{E}^3_\infty$) then to each point x not on C_0 (S_0) there exists one and only one point $y \neq x$ such that a circle (a circle or a sphere) through x passes through y if and only if it intersects C_0 (S_0) orthogonally.

If C_0 is a line in $\mathbb{E}^2$ (S_0 is a plane in $\mathbb{E}^3$), clearly the point y in question is the image of x under the reflection in C_0 (S_0). Assume that C_0 is an ordinary circle (S_0 is an ordinary sphere) with centre c and radius ϱ. If $x = \infty$, then $y = c$ has the properties required, and conversely, if $x = c$, then $y = \infty$. If $x \neq c, \infty$, then y must lie

on the half-line from c through x, and because of (1) we therefore have

$$y = c + \varrho^2 |x - c|^{-2} (x - c) . \tag{2}$$

If the right-hand side is interpreted as c for $x = \infty$ and as ∞ for $x = c$, (2) determines an involutory transformation of $\mathbb{E}^2_\infty$ ($\mathbb{E}^3_\infty$). It is called the *inversion in* C_0 (S_0) and denoted by inv_{C_0} (inv_{S_0}). If C_0 is a line (S_0 is a plane), inv_{C_0} (inv_{S_0}) is by definition the reflection in this line (plane). According to these definitions, C_0 (S_0) is pointwise fixed. In all cases the inversion is easily seen to be a homeomorphism.

Since a similarity φ maps circles onto circles (spheres onto spheres) and preserves orthogonality, we have

$$\begin{aligned} \varphi \circ \mathrm{inv}_{C_0} \circ \varphi^{-1} &= \mathrm{inv}_{\varphi(C_0)} \\ \varphi \circ \mathrm{inv}_{S_0} \circ \varphi^{-1} &= \mathrm{inv}_{\varphi(S_0)} . \end{aligned} \tag{3}$$

In particular, every inversion in a circle in $\mathbb{E}^2$ (a sphere in $\mathbb{E}^3$) can be transformed by a similarity into the inversion in the unit circle (the unit sphere). This will simply be denoted by inv.

An inversion of $\mathbb{E}^2_\infty$ (of $\mathbb{E}^3_\infty$) maps circles onto circles (spheres onto spheres and thus circles onto circles).

For an inversion in a line (in a plane) this is obvious. By the previous remark it is, therefore, sufficient to consider the inversion inv in the unit circle (unit sphere), that is

$$y = |x|^{-2} x . \tag{4}$$

Any circle in $\mathbb{E}^2_\infty$ (sphere in $\mathbb{E}^3_\infty$) has an equation of the form

$$\alpha |x|^2 - 2 < b, x > + \gamma = 0 , \tag{5}$$

where

$$\alpha, \gamma \in \mathbb{R} , \quad b \in \mathbb{E}^2 (\in \mathbb{E}^3) , \quad |b|^2 - \alpha\gamma > 0 ,$$

with the convention that $x = \infty$ satisfies the equation if and only if $\alpha = 0$. Indeed, if $\alpha = 0$, (5) is an equation for the line (plane) normal to the vector b through the point $\frac{1}{2}\gamma b/|b|^2$, otherwise (5) is an equation for the ordinary circle (sphere) with centre b/α and radius $(|b|^2 - \alpha\gamma)^{1/2}/|\alpha|$. Now, if x satisfies (5), then $y = x/|x|^2$ satisfies an equation equivalent to

$$\alpha - 2 < b, y > + \gamma |y|^2 = 0 .$$

and conversely. This proves the statement.

The following properties of inv_{C_0} (inv_{S_0}) are immediate consequences of the definition: Any circle (sphere or circle) intersecting C_0 (S_0) orthogonally is mapped onto itself. The two disks bounded by C_0 (the two balls bounded by S_0)

are interchanged, and hence inv_{C_0}) reverses orientation. Circles (spheres) which touch each other are mapped onto circles (spheres) which touch each other. (For lines and planes this means that they are parallel.)

The inversions in $\mathbb{E}^2_\infty$ preserve the absolute value of the angular measure and reverse its sign. The inversions in $\mathbb{E}^3$ preserve the angular measure. For reflections in a line (plane) this is obvious. Let C_0 (S_0) be an ordinary circle (sphere) and x any point not on it, and let $y = \mathrm{inv}_{C_0}(x)$ $(y = \mathrm{inv}_{S_0}(x))$. Given any two non-zero vectors at x, there are two circles through x and y, hence intersecting C_0 (S_0) orthogonally, each touching one of the vectors. Now x is mapped into y and each of the circles onto itself with sense reversed. This proves the statement for the complement of C_0 (of S_0). However, as observed previously, inv_{C_0} (inv_{S_0}) is the transform by a similarity of the inversion in a circle (sphere) which can be chosen disjoint from C_0 (S_0). This completes the proof of the statement.
In the sequel mostly the inversion in the unit sphere and its restriction to $\mathbb{E}^2_\infty$, that is, the inversion in the unit circle will play a role. It will be convenient to consider its composition with the reflection in the plane through the origin spanned by e_2 and e_3. This *anti-inversion* (as well as its restriction to $\mathbb{E}^2_\infty$) will be denoted by ω, thus

$$\omega(x_1, x_2, x_3) = (-x_1, x_2, x_3)\,(x_1^2 + x_2^2 + x_3^2)^{-1}$$

with the conventions

$$\omega(0,0,0) = \infty\,, \quad \omega(\infty) = (0,0,0)\,.$$

Clearly, ω is an involutory, orientation-preserving and conformal transformation.

II.4 Circle- and sphere-preserving transformations

The group of all transformations of $\mathbb{E}^2_\infty$ which map circles onto circles, the *Möbius group of* $\mathbb{E}^2_\infty$, will be denoted by $\mathscr{M}_2$. The group of all transformations of $\mathbb{E}^3_\infty$ which map spheres onto spheres, and thus circles onto circles, the *Möbius group of* $\mathbb{E}^3_\infty$, will be denoted by $\mathscr{M}_3$.

Since two points of $\mathbb{E}^2_\infty$ belong to the same disk, if and only if there is a circle through them which lies entirely in the disk, the elements of $\mathscr{M}_2$ map disks and, hence, circular flags onto circular flags. Similarly it is seen that the elements of $\mathscr{M}_3$ map caps onto caps, balls onto balls, and thus spherical flags onto spherical flags.

Given two circular flags (spherical flags) F' and F in $\mathbb{E}^2_\infty$ (in $\mathbb{E}^3_\infty$), there exists one and only one element ψ of $\mathscr{M}_2$ (of $\mathscr{M}_3$) such that $\psi(F') = F$.

To prove this it is obviously sufficient to consider the case where F' is the basic flag F_0. If the first point p of F is ∞, thus F a flag in $\mathbb{E}^2$ (in $\mathbb{E}^3$), then a transform-

ation ψ satisfying the requirements is, as shown in II.1, a similarity and uniquely determined. If $p \neq \infty$, let τ denote the translation $x \mapsto x + p$. Then $\omega \circ \tau^{-1}$, where ω is the anti-inversion, maps F onto a flag with the first point $\omega \circ \tau^{-1}(p) = \infty$. According to II.1, there is one and only one circle-preserving (sphere-preserving) transformation, namely a similarity φ, mapping F_0 onto this flag. Hence

$$\psi = (\omega \circ \tau^{-1})^{-1} \circ \varphi = \tau \circ \omega \circ \varphi,$$

and only this transformation satisfies the requirements.

Since a circular flag in $\mathbb{E}^2_\infty$ can be extended to a spherical flag in $\mathbb{E}^3_\infty$ in exactly two ways, namely by adjoining one of the half-spaces bounded by $\mathbb{E}^2_\infty$, we have the corollary:

Every element of $\mathcal{M}_2$ can be extended uniquely to an element of $\mathcal{M}_3$ which maps each of the half-spaces bounded by $\mathbb{E}^2_\infty$ onto itself.

The argument above shows that every element of $\mathcal{M}_2$ (of $\mathcal{M}_3$) can be obtained by composing similarities and the anti-inversion. Since the transformations are conformal, we have:

The elements of $\mathcal{M}_2$ (of $\mathcal{M}_3$) are conformal transformations.

A direct similarity has a unique representation $\varphi = \sigma \circ \varphi_0$, where σ is a translation and φ_0 a direct similarity leaving the origin o fixed. Let ϱ denote the reflection in the plane through o spanned by e_1 and e_3, that is

$$\varrho : (x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3).$$

Every opposite similarity has then a unique representation of the form $\varphi = \sigma \circ \varphi_0 \circ \varrho$.

Denoting by $\mathcal{M}_2^+$ (by $\mathcal{M}_3^+$) the subgroup of $\mathcal{M}_2$ (of $\mathcal{M}_3$) consisting of the orientation preserving elements and by $\mathcal{M}_2^-$ (by $\mathcal{M}_3^-$) its coset, we can summarize:

Every element ψ of $\mathcal{M}_2^+$ (of $\mathcal{M}_3^+$) has a unique representation

$$\begin{aligned} &\psi = \sigma \circ \varphi_0 && \text{if} \quad \psi(\infty) = \infty, \\ &\psi = \tau \circ \omega \circ \sigma \circ \varphi_0 && \text{if} \quad \psi(\infty) \neq \infty. \end{aligned}$$

Every element ψ of $\mathcal{M}_2^-$ (of $\mathcal{M}_3^-$) has a unique representation

$$\begin{aligned} &\psi = \sigma \circ \varphi_0 \circ \varrho && \text{if} \quad \psi(\infty) = \infty, \\ &\psi = \tau \circ \omega \circ \sigma \circ \varphi_0 \circ \varrho && \text{if} \quad \psi(\infty) \neq \infty. \end{aligned}$$

Here σ and τ are translations and φ_0 a direct similarity leaving o fixed, all belonging to $\mathcal{M}_2^+$ (to $\mathcal{M}_3^+$), ω the anti-inversion in the unit sphere and ϱ the reflection in the plane through o spanned by e_1 and e_3, in the case of $\mathcal{M}_2$ their restrictions to $\mathbb{E}^2_\infty$.

We notice further that if $\psi \in \mathcal{M}_3$ maps each of the half-spaces bounded by $\mathbb{E}^2_\infty$ onto itself, then each of the composing transformations does the same. For ω and

ϱ this is clearly the case, for the translations because their restrictions to $\mathbb{E}^2_\infty$ must belong to $\mathcal{M}_2$ and consequently also for the similarity φ_0.

As a consequence of the fact that there is one and only one element of $\mathcal{M}_2$ which maps a given circular flag in $\mathbb{E}^2_\infty$ onto another such flag we note:

Given three distinct points p, q, r and three distinct points p', q', r', all in $\mathbb{E}^2$, there is one and only one element $\psi \in \mathcal{M}_2^+$ and one and only one element $\psi^* \in \mathcal{M}_2^-$ such that

$$\begin{aligned} \psi(p) &= p', & \psi(q) &= q', & \psi(r) &= r', \\ \psi^*(p) &= p', & \psi^*(q) &= q', & \psi^*(r) &= r'. \end{aligned}$$

II.5 The Möbius group of the upper half-space

In the following we are dealing with the subgroup of $\mathcal{M}_3$ whose elements map a ball B, and thus also its bounding sphere, onto itself. Since there exist elements of $\mathcal{M}_3$ which map a ball onto any other, two such subgroups of $\mathcal{M}_3$ are conjugate. Therefore it is sufficient to consider one of them. For formal reasons it is convenient to choose for B the *upper half-space*

$$U = \{(x_1, x_2, x_3) \in \mathbb{E}^3 \mid x_3 > 0\}.$$

Let $\mathcal{H}_3$ denote the subgroup leaving it invariant, $\mathcal{H}_3^+$ its subgroup consisting of direct transformations and $\mathcal{H}_3^-$ the coset of the latter. As shown in the preceding section, their restrictions to $\mathbb{E}^2_\infty$ are precisely $\mathcal{M}_2$, $\mathcal{M}_2^+$, and $\mathcal{M}_2^-$.

The group $\mathcal{H}_3$ has a simple representation in terms of $\mathbb{J}$-quaternions (cf. I.1). To derive it, we identify $\mathbb{E}^3_\infty$ with the extended space of $\mathbb{J}$-quaternions, $\mathbb{J}_\infty$, by assigning to the point (x_1, x_2, x_3) the quaternion $x_1 + x_2 i + x_3 j$. Writing x instead of $x_1 + x_2 i$ and ξ instead of x_3, we represent the points x by the quaternion

$$\mathrm{x} = x + \xi j, \quad x \in \mathbb{C}, \quad \xi \in \mathbb{R}.$$

For every number $a \in \mathbb{C} \setminus \{0\}$ the map

$$\mathrm{x} \mapsto a\mathrm{x}a = a^2 x + a\bar{a}\xi j, \quad \mathrm{x} \in \mathbb{J}_\infty,$$

is a direct similarity belonging to $\mathcal{H}_3^+$ and leaving o fixed. Indeed, it is the rotation about the j-axis through the angle $2 \arg a$ composed with the dilatation with centre o and ratio $|a|^2$. Clearly, every direct similarity belonging to $\mathcal{H}_3^+$ and leaving o fixed can be obtained in this manner, and a is determined up to replacement by $-a$.

For every $b \in \mathbb{C}$ the map

$$\mathrm{x} \mapsto \mathrm{x} + b = x + b + \xi j, \quad \mathrm{x} \in \mathbb{J}_\infty,$$

is a translation belonging to $\mathscr{H}_3^+$. Every such translation can be obtained in this manner, and b is uniquely determined.

The anti-inversion ω may be written

$$\mathrm{x} \mapsto -\mathrm{x}^{-1} = (-\bar{x} + \xi j)/(x\bar{x} + \xi^2), \quad \mathrm{x} \in \mathbb{J}_\infty,$$

and the reflection ϱ

$$\mathrm{x} \mapsto -j\mathrm{x}j = \bar{x} + \xi j, \qquad \mathrm{x} \in \mathbb{J}_\infty.$$

Together with the last result of II.4 these observations yield:

Every element ψ of $\mathscr{H}_3^+$ can be written

$$(1) \qquad \begin{aligned} &\psi(\mathrm{x}) = a\mathrm{x}a + b && \text{if } \psi(\infty) = \infty, \\ &\psi(\mathrm{x}) = -(a\mathrm{x}a + b)^{-1} + c && \text{if } \psi(\infty) \neq \infty. \end{aligned}$$

Every element ψ of $\mathscr{H}_3^-$ can be written

$$\begin{aligned} &\psi(\mathrm{x}) = -aj\mathrm{x}ja + b && \text{if } \psi(\infty) = \infty, \\ &\psi(\mathrm{x}) = -(-aj\mathrm{x}ja + b)^{-1} + c && \text{if } \psi(\infty) \neq \infty. \end{aligned}$$

Here a, b, $c \in \mathbb{C}$, $a \neq 0$, and these numbers are uniquely determined by ψ up to replacement of a by $-a$.

This result leads to a representation of the elements of $\mathscr{H}_3$ as fractional linear transformations. With a matrix

$$\mathbf{f} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \quad f_{\lambda\mu} \in \mathbb{C}, \quad \det \mathbf{f} = 1,$$

we associate the mapping $f\colon \mathbb{J}_\infty \to \mathbb{H}_\infty$ defined by

$$(3) \qquad f(\mathrm{x}) = (f_{11}\mathrm{x} + f_{12})(f_{21}\mathrm{x} + f_{22})^{-1}$$

with the convention

$$f(\infty) = f_{11}/f_{21}$$

(understood tacitly in the sequel). It is easily verified that if f and g correspond to the matrices $\mathbf{f}$ and $\mathbf{g}$, respectively, $g \circ f$ corresponds to the matrix $\mathbf{gf}$.

The mappings

$$\mathrm{x} \mapsto a\mathrm{x}a, \quad \mathrm{x} \mapsto \mathrm{x} + b, \quad \mathrm{x} \mapsto -\mathrm{x}^{-1}$$

correspond to the matrices

$$(4) \qquad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence the transformations (1) correspond to the matrices

$$\text{(5)} \qquad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & ba^{-1} \\ 0 & a^{-1} \end{pmatrix},$$

$$\text{(6)} \qquad \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} ac & (cb-1)a^{-1} \\ a & ba^{-1} \end{pmatrix}.$$

As stated in I.4, every matrix $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ may be obtained in this way. Hence every element of $\mathscr{H}_3^+$ has a representation (3), and conversely. The transformation (3) is the identity if and only if $\mathbf{f} = \pm \mathbf{1}$.

Since SL $(2, \mathbb{C})$, as observed in I.4, is generated by the matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in \mathbb{C}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$\mathscr{H}_3^+$ is generated by the translations $\mathrm{x} \mapsto \mathrm{x} + b$ and the anti-inversion $\mathrm{x} \mapsto -\mathrm{x}^{-1}$.

Summarizing we can state:

The correspondence of the transformations

$$f(\mathrm{x}) = (f_{11}\mathrm{x} + f_{12})(f_{21}\mathrm{x} + f_{22})^{-1}, \quad \mathrm{x} \in \mathbb{J}_\infty,$$

to the matrices $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ is a homomorphism of $\mathrm{SL}(2, \mathbb{C})$ onto $\mathscr{H}_3^+$ with kernel $\{\mathbf{1}, -\mathbf{1}\}$ and thus $\mathscr{H}_3^+$ isomorphic with $\mathrm{PSL}(2, \mathbb{C})$.

With the restriction $\mathrm{x} \in \mathbb{C}_\infty$ this holds for $\mathscr{M}_2^+$ instead of $\mathscr{H}_3^+$.

Clearly, the map (3) remains unchanged when the matrix $\mathbf{f}$ is multiplied by a non-zero real number. However, in contrast to the case of $\mathscr{M}_2^+$, a non-real complex factor is not permitted. To see this, consider the map $\mathrm{x} \mapsto u\mathrm{x}u^{-1}$ corresponding to the matrix $u\,\mathbf{1}$, $u \in \mathbb{C} \setminus \{0\}$. It maps j onto $uju^{-1} = u\bar{u}^{-1}j$, which belongs to the upper half-space U only if $u\bar{u}^{-1} = 1$.

We mention that if and only if $\det \mathbf{f} \in \mathbb{R}$, one has

$$(f_{11}\mathrm{x} + f_{12})(f_{21}\mathrm{x} + f_{22})^{-1} = (\mathrm{x}f_{21} + f_{22})^{-1}(\mathrm{x}f_{11} + f_{12}) \quad \text{for} \quad \mathrm{x} \in \mathbb{J}_\infty.$$

We turn now to $\mathscr{H}_3^-$. The reflection

$$\mathrm{x} \mapsto -j\mathrm{x}j$$

may be associated with the matrix

$$\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} = \mathbf{1}j,$$

hence any element of $\mathscr{H}_3^-$ with a matrix $\mathbf{f}j$ where $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$. Observing that

$$\mathbf{f}j = j\bar{\mathbf{f}},$$

and

$$(\mathbf{f}j)^{-1} = -j\mathbf{f}^{-1} = -\bar{\mathbf{f}}^{-1}j,$$

one easily verifies that multiplication of such matrices obeys the usual rules and that the matrices of $\mathrm{SL}(2, \mathbb{C}) \cup \mathrm{SL}(2, \mathbb{C})j$ form a group. (However, the rules for transposition and the operation $\sim$ defined in I.3 are in general not valid.) One infers:

By letting correspond the transformations $\mathrm{x} \to f(\mathrm{x})$ to the matrices $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ and the transformations $\mathrm{x} \mapsto f(-j\mathrm{x}j)$ to the matrices $\mathbf{f}j \in SL(2, \mathbb{C})j$, a homomorphism of $\mathrm{SL}(2, \mathbb{C}) \cup \mathrm{SL}(2, \mathbb{C})j$ onto $\mathscr{H}_3$ with kernel $\{\mathbf{1}, -\mathbf{1}\}$ is obtained.

Notes to Chapter II

Sections 1–4 deal with well-known topics of elementary analytic geometry which are to be found in various text books.

The representation of the Möbius group of the upper half-space as the group of fractional linear transformations of a $\mathbb{J}$-quaternion variable, dealt with in Section 5, was stated in a more general context by Vahlen [31]. A direct proof is due to Fueter [9]. Cf. also Gormley [10] and Szybiak [30].

III. The Basic Notions of Hyperbolic Geometry

III.1 Lines and planes. Convexity

We provide the upper half-space U with a geometrical structure which makes it a model of *hyperbolic 3-space*.

By definition, an *h-line* is the semicircle contained in U of a circle in $\mathbb{J}_\infty$ which intersects the boundary $\mathbb{C}_\infty$ of U orthogonally. (It is recalled that the lines orthogonal to $\mathbb{C}_\infty$ are included.)

An *h-plane* is the hemisphere contained in U of a sphere in $\mathbb{J}_\infty$ which intersects $\mathbb{C}_\infty$ orthogonally. (The planes orthogonal to C are included.) Clearly, the elements of $\mathscr{H}_3$ map h-lines onto h-lines and h-planes onto h-planes.

Though the space of the geometry to be developed is U, it is frequently convenient to take the points of its boundary $\mathbb{C}_\infty$ into consideration. These points will be called the *improper points*, those of U the *proper points* whenever doubt may arise. Let $\operatorname{cl} U = U \cup \mathbb{C}_\infty$.

Every h-line has two distinct improper endpoints, briefly called its *ends*. Any two distinct points of $\mathbb{C}_\infty$ are the ends of precisely one h-line. An h-line may be *oriented* by choosing one of the ends as the *initial point*. The other one will then be called the *terminal point*. The oriented line with initial point u and terminal point u' will be denoted by $[u, u']$.

The improper boundary of an h-plane is a circle in $\mathbb{C}_\infty$ which will be called its *horizon*. Every circle in $\mathbb{C}_\infty$ is the horizon of precisely one h-plane. An h-plane divides U into two *h-half-spaces*. Each of them has the points of one of the disks bounded by the horizon as improper points. An h-plane may be oriented by assigning the sign $+$ to one of the h-half-spaces or, equivalently, to the corresponding disk and the sign $-$ to the other h-half-space or disk.

By adjoining the upper half-space U to a circular flag in $\mathbb{C}_\infty$ a spherical flag in $\mathbb{J}_\infty$ is obtained (cf. II.2). Given any two such spherical flags, there exists one and only one element of $\mathscr{H}_3$ which maps one onto the other (cf. II.4).

A corollary to be used frequently is:

Given any two oriented h-lines (h-planes), there are elements of $\mathscr{H}_3$ which map one onto the other.

The following incidence relations between points, h-lines, and h-planes are easily proved using elementary Euclidean geometry:

Given any two distinct (proper or improper) points, there exists one and only one h-line containing them.

Given any three non-h-collinear (proper or improper) points, there exists one and only one h-plane containing them.

If an h-line and an h-plane have two distinct (proper or improper) points in common, then the h-line is contained in the h-plane.

If two distinct h-planes have a proper point in common, then their intersection is an h-line.

Concerning the mutual positions of h-planes and h-lines, the following terminology will be used.

Two distinct h-planes *intersect*, are *parallel*, or are *ultraparallel* according as their horizons intersect in two distinct points, are tangent, or are disjoint.

An h-plane and an h-line not contained in it *intersect*, are *parallel*, or are *ultraparallel*, according as the ends of the h-lines are separated by the horizon of the h-plane, one of the ends belongs to the horizon, or the ends are not separated by the horizon.

The distinct h-lines $[u, u']$ and $[v, v']$ are *h-coplanar* if and only if the four ends lie on a circle, equivalently, if and only if the cross ratio $\not{R}(u, u', v, v') \in \mathbb{R}_\infty$. In that case, the h-lines *intersect*, are *concurrent*, if the point pairs (u, u') and (v, v') separate each other on that circle, thus, if $\not{R}(u, u', v, v') < 0$. The h-lines are *parallel*, if the pairs (u, u') and (v, v') have a point in common, thus, if $\not{R}(u, u', v, v') = 0$ or $= \infty$. The h-lines are *ultraparallel*, if the point pairs (u, u') and (v, v') have no point in common and do not separate each other on the circle containing them, thus, if $\not{R}(u, u', v, v') > 0$.

Two h-lines which are not h-coplanar are said to be *skew*.

Using the topology of $\mathbb{J}_\infty$ and the usual order relation for points on semicircles and half-lines, we may speak of closed, open, half-open h-segments and h-half-lines.

This permits to define h-convexity in the usual manner.

A subset C of cl U is *h-convex* if for any two (proper or improper) points of C the h-segment, h-half-line, or h-line joining these points is contained in C.

Clearly, the intersection of h-convex sets is h-convex. Therefore the *h-convex hull* of an arbitrary subset of cl U may be defined as the intersection of all h-convex sets containing it.

An h-plane is said to *support* a subset M of cl U if it contains at least one point of cl M and M is contained in one of the closed h-half-spaces bounded by it.

Suppose that M is a subset of an h-plane P. Then an h-line in P is said to *support* M if it contains at least one point of cl M and M is contained in one of the closed h-half-planes of P bounded by this h-line.

Through every proper boundary point of a convex subset of cl U there is a supporting h-plane.

Let C be a convex subset of an h-plane H. Then through every proper boundary point of C relative to H there is a supporting h-line.

A closed convex subset of cl U is the intersection of all closed h-half-spaces

containing it. It is sufficient to consider the h-half-spaces bounded by supporting h-planes. The same holds for relatively closed convex subsets of U, provided the improper points of the h-half-spaces are excluded.

Analogous statements are valid for convex subsets of h-planes.

All of the statements about convexity and, by the way, also the previous ones about incidence, are immediate consequences of their Euclidean counterparts. This is seen by mapping $\operatorname{cl} U$ onto a "projective model" of hyperbolic 3-space.

Let $\mathbb{E}^3$ be a Euclidean 3-space in which an orthonormal coordinate system has been chosen. Define

$$\Phi : \operatorname{cl} U \to \mathbb{E}^3$$

by $x + \xi j \mapsto (\eta_1, \eta_2, \eta_3)$, where, for $x + \xi j \neq \infty$,

$$\eta_1 = \frac{2\operatorname{Re} x}{1+|x|^2+\xi^2}, \quad \eta_2 = \frac{2\operatorname{Im} x}{1+|x|^2+\xi^2}, \quad \eta_3 = \frac{1-|x|^2-\xi^2}{1+|x|^2+\xi^2}.$$

and $\qquad \eta_1 = 0, \quad \eta_2 = 0, \quad \eta_3 = -1 \quad \text{for} \quad x + \xi j = \infty$. Since

$$\eta_1^2 + \eta_2^2 + \eta_3^2 = 1 - \frac{4\xi^2}{(1+|x|^2+\xi^2)^2} \leqq 1$$

and Φ has the inverse

$$x = \frac{\eta_1 + \eta_2 i}{1+\eta_3}, \quad \xi = \frac{(1-\eta_1^2-\eta_2^2-\eta_3^2)^{1/2}}{1+\eta_3} \geqq 0$$

for $\eta_1^2 + \eta_2^2 + \eta_3^3 \leqq 1$, it follows that Φ is a homeomorphism of $\operatorname{cl} U$ onto the closed unit ball.

Every h-plane satisfies an equation

$$\alpha(|x|^2 + \xi^2) - b\bar{x} - \bar{b}x + \gamma = 0,$$

where $\alpha, \gamma \in \mathbb{R}$, $b \in \mathbb{C}$, $|b|^2 - \alpha\gamma > 0$. Because of

$$|x|^2 + \xi^2 = \frac{1-\eta_3}{1+\eta_3}$$

its image satisfies

$$\alpha(1-\eta_3) - 2\operatorname{Re} b \cdot \eta_1 - 2\operatorname{Im} b \cdot \eta_2 + \gamma(1+\eta_3) = 0$$

and thus is the intersection of a plane with the unit ball. Conversely, any plane intersecting the unit ball has an equation which can be written in this form with $|b|^2 > \alpha\gamma$. The images of the h-planes are consequently all the disks inscribed in the unit sphere and the images of the h-lines therefore its chords.

III.2 Orthogonality

In cl $U \subset \mathbb{J}_\infty = \mathbb{E}^3_\infty$ we shall use the angular measure previously introduced (cf. II.2).

Since the following statements are invariant under the transformations belonging to the group $\mathscr{H}_3$, the assumptions in some of the proofs that one of the h-planes or h-lines occurring is a vertical half-plane or half-line are permissible.

Two h-planes intersect orthogonally, are *normal* to each other, if and only if their horizons intersect orthogonally.

An h-plane P and an h-line L intersect orthogonally, L is a *normal* to P, if and only if the ends of L are interchanged by the inversion in the horizon of P. Recalling the definition of the inversion (II.3), this is seen to be equivalent to: at least two distinct h-planes through L, and then all of them are normal to P.

Two h-lines $[u, u']$ and $[v, v']$ intersect orthogonally, are *normals* of each other, if and only if the point pairs (u, u') and (v, v') are harmonic.

To see this, assume that $[u, u'] = [0, \infty]$. Then $\not{R}(u, u', v, v') = \not{R}(0, \infty, v, v') = -1$ is equivalent to $v' = -v$. On the other hand, an h-line $[v, v']$ intersecting $[0, \infty]$ orthogonally is a semi-circle with centre 0, and this is likewise equivalent to $v' = -v$.

It will be convenient to admit *improper h-lines* with coinciding ends. Geometrically an improper h-line $[u, u]$ is represented by the improper point u. In accordance with the convention to consider two point pairs to be harmonic if three of the points coincide, an improper h-line $[v, v]$ will be called a normal to the h-line $[u, u']$ if $v = u$ or $v = u'$. To avoid exceptions in some statements we admit here that $u = u'$, thus an improper h-line will be considered a normal to itself.

Furthermore we introduce *improper h-planes* the horizons of which are "point circles", and hence they are likewise represented by improper points. An improper h-plane will be considered as normal to itself and to every proper h-plane the horizon of which passes through it. An h-line will be considered to be normal to an improper h-plane if one or both of its ends coincide with the latter.

In the following statements the terms "point", "h-line" and "h-plane", if not specified, stand for proper and improper elements. Whether a point of $\mathbb{C}_\infty$ has to be considered as an improper point, h-line or h-plane will be clear from the context.

2.1 For any h-plane P and any point p there is an h-line normal to P and containing p. It is unique unless P and p both are improper and coincide.

If P is improper or p on the horizon of P, the statement follows from the conventions above. Otherwise assume P to be a vertical half-plane. Then the h-line in question is the semi-circle in cl U which lies in the plane through p orthogonal to P, has its centre on the horizon of P, and contains p.

2.2 For any h-line L and any point p there is an h-plane normal to L and containing p. It is unique unless L is improper.

If L is improper, any h-plane through the h-line joining p and L satisfies the requirement. If p is improper and coincides with an end of L, the improper h-plane coinciding with p is the only one normal to L and containing p. In all other cases we may assume that $L = [0, \infty]$ and, thus, $p \neq 0, \infty$. Then the h-plane in question is the hemisphere in $\operatorname{cl} U$ with centre 0 and containing p.

2.3 For any h-line L and any point p there is an h-line normal to L and containing p. It is unique unless L is proper and p a proper point of it. In this case the h-lines through p in the h-plane intersecting L orthogonally at p, and no others, satisfy the requirement.

If L is improper, $[u, u]$ say, the h-line joining u and p (improper if $p = u$) and no other satisfies the requirement. If L is proper, there exists, according to the preceding statement, one and only one h-plane normal to L and containing p. The h-line joining p and its intersection with L, provided these points are distinct, otherwise the h-lines through p in this h-plane, and no others are the normals in question.

2.4 Any two distinct non-intersecting h-planes have a unique common normal.

Denote the h-planes by P and P'. If one of them or both are improper, the statement is obvious. If both are proper and parallel, the common point of their horizons represents the improper common normal, and there are no others. If P and P' are proper and ultraparallel, assume P to be a vertical half-plane. Then P' is a hemisphere disjoint with P. An h-line normal to P is a semi-circle with centre on the horizon of P and contained in a plane Q orthogonal to P. That the semi-circle intersects P' orthogonally requires that this plane Q passes through the centre of P' since the endpoints of the semi-circle have to be inverse with respect to the horizon of P'. The unique semi-circle with the properties desired has its centre at the point where Q intersects the horizon of P and its radius equals the tangent from this point of the semi-circle in which Q intersects P'.

2.5 For any h-plane P and any h-line L there is an h-plane containing L and normal to P. It is unique unless L is improper or normal to P.

If L is improper, the horizons of the h-planes in question are the circles through L and orthogonal to the horizon of P provided P is proper, otherwise the circles through P and L. If L is proper and normal to P, in particular if P is improper and coincides with an end of L, all h-planes through L are normal to P. In all other cases the ends of L are not inverse with respect to the horizon of P if P is proper and are different from P if P is improper. The statement is then that there is one and only one circle through the ends of L and orthogonal to or through the horizon of P. This is obviously true if P is improper. Otherwise the circle through the ends of L and the inverse of one of them with respect to the horizon of P and no other, has the properties requires.

2.6 Any two distinct h-lines have a unique common normal.

Let the h-lines be $[u, u']$ and $[v, v']$. If one of them is improper, say $v' = v$, and coincides with an end of the other one, $[v, v]$ is a common normal and the only one. In all other cases no three of the points u, u', v, v' coincide andthe unordered pairs (u, u') and (v, v') are different. There exists therefore one and only one point pair harmonic with both of them (cf. I.4). This proves the statement.

2.7 Any h-plane and any h-line which have no proper point in common have a common normal. It is unique unless both are improper and coincide.

Let P denote the h-plane and L the h-line. The case where L is improper is covered by 2.1 and the case where P is improper by 2.3. If P and L are proper, they are not normal to each other by assumption, and by 2.5 there is one and only one h-plane through L orthogonal to P. A common normal of P and L must lie in this h-plane. Consequently it is the common normal of the latter's intersection with P and L. This normal exists and is unique by 2.6.

The purpose of the following is to discuss the mutual positions of three distinct h-planes.

A set of h-planes is called a *pencil* if it satisfies one of the following conditions:

1) It consists of all h-planes which contain a proper h-line, equivalently, the horizons of which pass through two distinct improper points. (*Elliptic pencil.*)

2) It consists of all h-planes which are mutually parallel such that their horizons touch each other at the same improper point. The latter is considered as an improper h-plane belonging to the pencil. (*Parabolic pencil.*)

3) It consists of all h-planes which intersect a proper h-line orthogonally, equivalently, such that two distinct inproper points are inverse with respect to the horizon of each of them. These points are considered as improper h-planes belonging to the pencil. (*Hyperbolic pencil.*)

Any two distinct h-planes belong to one and only one pencil. Clearly, this is elliptic, parabolic, or hyperbolic according as the two h-planes intersect, are parallel, or are ultraparallel.

2.8 Three h-planes not belonging to the same pencil either have one and only one proper or improper point in common, or there is one and only one proper h-plane orthogonal to them.

Let P_1, P_2, P_3 denote the h-planes. Assume first that two of them, P_1 and P_2 say, are proper and intersect. By assumption the h-line L of intersection is not contained in P_3. If it intersects or is parallel to P_3, the common proper or improper point and no other is common to P_1, P_2, P_3. If L and P_3 are ultraparallel, there is, according to 2.7, a unique common normal of L and P_3. The h-plane which contains this normal and is orthogonal to L, and no other is orthogonal to P_1, P_2, and P_3.

Assume now that P_1 and P_2 are proper or improper and ultraparallel if both are proper. By assumption the common normal N of P_1 and P_2 is not orthogonal to P_3. Hence, by 2.5, there is a unique h-plane containing N and orthogonal to P_3.

This h-plane and no other is orthogonal to P_1, P_2, P_3.

A set of h-planes is called a *bundle* if it satisfies one of the following conditions:

1) It consists of all h-planes passing through a proper point. (*Elliptic bundle.*)

2) It consists of all h-planes the horizons of which contain the same improper point. The latter is considered as an improper h-plane belonging to the bundle. (*Parabolic bundle.*)

3) It consists of all h-planes orthogonal to a proper h-plane. The points of the horizon of the latter are considered as improper h-planes belonging to the bundle. (*Hyperbolic bundle.*)

Any three h-planes not belonging to the same pencil belong to one and only one bundle. This is elliptic, parabolic, or hyperbolic according to the three cases in 2.8. The pencil determined by two h-planes of a bundle is contained in the latter.

An *orthogonal frame* is by definition an ordered triple $([u_1, u'_1], [u_2, u'_2], [u_3, u'_3])$ of proper, oriented, concurrent h-lines which are mutually orthogonal.

2.9 Given any two orthogonal frames, there is a unique element of the group $\mathscr{H}_3$ which maps one onto the other. This element belongs to $\mathscr{H}_3^+$ or $\mathscr{H}_3^-$ according as the frames determine the same or opposite orientations of the space.

To see this we associate with each of the frames a spherical flag in the following manner. Let $([u_1, u'_1], [u_2, u'_2], [u_3, u'_2])$ be the first frame and define the flag F_u to consist of the points u_1, u'_1, u_2 in this order, that disk bounded by the circle through them which contains u_3, and the upper half-space U. For the second frame $([v_1, v'_1], [v_2, v'_2], [v_3, v'_3])$ define F_v analogously. As shown in II.4, there is one and only one element f of $\mathscr{M}_3$ which maps F_u onto F_v, but this element belongs to $\mathscr{H}_3$ since it maps U onto itself. Since the restriction of f to $\mathbb{C}_\infty$ preserves real cross ratios and the pairs (u_1, u'_1), (u_2, u'_2) as well as the pairs (v_1, v'_1), (v_2, v'_2) are harmonic, $f(u_1) = v_1, f(u'_1) = v'_1, f(u_2) = v_2$ imply $f(u'_2) = v'_2$. Further, (u_3, u'_3) is harmonic to both (u_1, u'_1) and (u_2, u'_2). Hence, $(f(u_3), f(u'_3))$ must be harmonic to both (v_1, v'_1) and (v_2, v'_2). Since there is only one such pair, namely (v_3, v'_3) and v_3 and v'_3 belong to different disks bounded by the circle through v_1, v'_1, v_2 (they are inverse with respect to it), we must have $f(u_3) = v_3, f(u'_2) = v'_2$, as to be shown.

The last statement follows from the fact that the orientations determined by the frames agree with those determined by the corresponding spherical flags (cf. II.2).

III.3 The invariant Riemannian metric

We shall prove:

There is one and, apart from a positive constant factor, only one Riemannian metric in the upper half-space U which is invariant under the group $\mathscr{H}_3$, namely

$$ds^2 = \frac{dx\,d\bar{x}}{\xi^2} = \frac{dx\,d\bar{x} + d\xi^2}{\xi^2}.$$

That this metric is invariant under the translations $y = x + b, b \in \mathbb{C}$, is obvious. For the anti-inversion

$$y = y + \eta j = -x^{-1} = -(x + \xi j)^{-1} = (-\bar{x} + \xi j)/(x\bar{x} + \xi^2)$$

we have

$$d(yx) = y\,dx + dy\,x = 0,$$

hence

$$dy = -y\,dx\,x^{-1} = x^{-1}\,dx\,x^{-1}.$$

Since $\eta = \xi/(x\bar{x})$, this yields

$$\begin{aligned}\frac{dy\,d\bar{y}}{\eta^2} &= x^{-1}\,dx\,x^{-1}\bar{x}^{-1}\,d\bar{x}\,\bar{x}^{-1}(x\bar{x})^2/\xi^2 \\ &= \frac{dx\,d\bar{x}}{\xi^2}\end{aligned}$$

because $x\bar{x}$ and $dx\,d\bar{x}$ commute with every quaternion.

For the reflection $y = y + \eta j = -\bar{x} + \xi j$ the invariance is obvious. Since $\mathscr{H}_3$ is generated by the transformations considered, the invariance of the metric follows.

That it is essentially the only one with the property, can be seen as follows. If we write $x = x_1 + x_2 i$ with $x_1, x_2 \in \mathbb{R}$, we have to consider a positive definite quadratic form ds^2 in dx_1, dx_2, and $d\xi$. Because of the invariance under the translations in $\mathscr{H}_3$ the coefficients can only depend on ξ. Since the restriction of $\mathscr{H}_3$ to a plane $\xi = \text{const.}$ contains all Euclidean isometries of this plane, ds^2 for $d\xi = 0$ must be proportional to the Euclidean metric, that is, of the form $\varphi(\xi)(dx_1^2 + dx_2^2) = \varphi(\xi)dx\,d\bar{x}$ with a positive function φ. Furthermore, the terms with $dx_1\,d\xi$ and $dx_2\,d\xi$ must vanish because $\mathscr{H}_3$ contains the half-turn $(x_1, x_2, \xi) \mapsto (-x_1, -x_2, \xi)$ under which their sum changes sign. Indeed, this implies that this sum, a linear form in dx_1 and dx_2, vanishes identically. Hence, we must have

$$ds^2 = \varphi(\xi)dx\,d\bar{x} + \psi(\xi)d\xi^2$$

with a positive function ψ. The invariance under the dilatations $x + \xi j \mapsto \alpha x + \alpha\xi j$ with $\alpha > 0$, which belong to $\mathscr{H}_3$, implies that both $\varphi(\xi)$ and $\psi(\xi)$ are proportional with ξ^{-2}. Apart from a positive constant factor we must therefore have

$$ds^2 = (dx\,d\bar{x} + \lambda d\xi^2)/\xi^2$$

with a positive constant λ. It remains to be shown that $\lambda = 1$. This will follow from the invariance under the anti-inversion $y + \eta j = -(x + \xi j)^{-1}$. It requires that

$$(dy\,d\bar{y} + \lambda d\eta^2)/\eta^2 = (dx\,d\bar{x} + \lambda d\xi^2)/\xi^2$$

or, since $(dx\,d\bar{x} + d\xi^2)/\xi^2$ is invariant,

$$(\lambda - 1)\,d\eta^2/\eta^2 = (\lambda - 1)\,d\xi^2/\xi^2 .$$

Now,

$$\eta = \xi/(x\bar{x} + \xi^2)$$

and thus

$$d\eta = \frac{(x\bar{x} + \xi^2)\,d\xi - \xi(x\,d\bar{x} + \bar{x}\,dx + 2\,\xi\,d\xi)}{(x\bar{x} + \xi^2)^2} .$$

If $\lambda \neq 1$, we should therefore have identically

$$[(x\bar{x} - \xi^2)\,d\xi - \xi(x\,d\bar{x} + \bar{x}\,dx)]^2 = (x\bar{x} + \xi^2)^2\,d\xi^2 ,$$

which is obviously wrong.

As mentioned, the Riemannian metric

$$k\,d\mathrm{x}\,d\bar{\mathrm{x}}/\xi^2$$

with any positive constant k is invariant under $\mathscr{H}_3$. For reasons which need not be discussed here, $-1/k$ is called the *curvature* of the space provided with this metric. For almost all what follows the value of k is irrelevant. If nothing else is stated it is chosen equal to 1.

III.4 The hyperbolic metric

Let

$$\mathrm{x}(\tau) = x(\tau) + \xi(\tau)j , \quad \alpha \leq \tau \leq \beta ,$$

be a continuous and piecewise continuously differentiable curve in the upper half-space U. Its *h-length* is defined, by means of the Riemannian metric introduced, to be

$$\lambda = \int_\alpha^\beta \left(\left| \frac{dx}{d\tau} \right|^2 + \left(\frac{d\xi}{d\tau} \right)^2 \right)^{1/2} d\tau/\xi .$$

We claim that λ is greater than the h-length of the h-segment joining $x(\alpha)$ and $x(\beta)$, unless the curve coincides with this h-segment.

To prove this we may suppose that $\mathrm{x}(\alpha) = j$ and $\mathrm{x}(\beta) = \varrho j$ for some $\varrho > 1$. Clearly,

$$\lambda \geq \int\limits_{\alpha}^{\beta} \left|\frac{d\xi}{d\tau}\right| d\tau/\xi \geq \int\limits_{\alpha}^{\beta} \frac{d\xi}{d\tau}\, d\tau/\xi = \log \varrho\,.$$

The last integral, and thus $\log \varrho$, is the h-length of the h-segment joining j and ϱj. Equality requires that $dx/d\tau = 0$, hence $x = 0$, and $d\xi/d\tau \geq 0$ for $\alpha \leq \tau \leq \beta$, as claimed.

If we define the h-distance $\delta(\mathrm{a},\mathrm{b})$ of two points a and b in U as the h-length of the h-segment joining them, U becomes a metric space. Indeed, the statement above implies the triangle inequality

$$\delta(\mathrm{a},\mathrm{b}) + \delta(\mathrm{b},\mathrm{c}) \geq \delta(\mathrm{a},\mathrm{c})\,,$$

and that equality holds if and only if b lies on the h-segment joining a and c.

Obviously, the elements of $\mathscr{H}_3$ are isometries. Given two pairs of points (a, b) and (a′, b′) in U with $\delta(\mathrm{a},\mathrm{b}) = \delta(\mathrm{a}',\mathrm{b}')$, there is an $f \in \mathscr{H}_3$ with $f(\mathrm{a}) = \mathrm{a}'$, $f(\mathrm{b}) = \mathrm{b}'$.

To obtain an expression for the h-distance $\delta = \delta(\mathrm{a},\mathrm{b})$ of points $\mathrm{a} = a + \alpha j$ and $\mathrm{b} = b + \beta j$, $\alpha, \beta > 0$, we observe that

$$\frac{|\mathrm{b}-\mathrm{a}|^2}{\alpha\beta} = \frac{|b-a|^2 + (\beta-\alpha)^2}{\alpha\beta}$$

is invariant under the transformations of $\mathscr{H}_3$. The invariance under the translations $\mathrm{x} \mapsto \mathrm{x} + q$, $q \in \mathbb{C}$, and under the reflection $\mathrm{x} \mapsto -\bar{\mathrm{x}}$ is obvious. To see it for the anti-inversion it has to be checked that the expression remains unchanged if a and b are replaced by

$$-\mathrm{a}^{-1} = \frac{-\bar{a} + \alpha j}{|\mathrm{a}|^2} \quad \text{and} \quad -\mathrm{b}^{-1} = \frac{-\bar{b} + \beta j}{|\mathrm{b}|^2}$$

and, thus, α and β by $\alpha/|\mathrm{a}|^2$ and $\beta/|\mathrm{b}|^2$, respectively. To find how the expression depends on δ we may suppose that $\mathrm{a} = j$ and $\mathrm{b} = \varrho j$ for some $\varrho > 1$. Then $\delta = \log \varrho$ and

$$\frac{|\mathrm{b}-\mathrm{a}|^2}{\alpha\beta} = \frac{(\varrho-1)^2}{\varrho} = \frac{(e^\delta-1)^2}{e^\delta} = 4\sinh^2\frac{\delta}{2}\,.$$

Hence we obtain

$$\text{(1)} \qquad \sinh^2\frac{\delta}{2} = \frac{|\mathrm{b}-\mathrm{a}|^2}{4\alpha\beta}$$

or, using $\cosh\delta = 2\sinh^2\frac{\delta}{2} + 1$,

$$\cosh\delta = \frac{|b-a|^2 + \alpha^2 + \beta^2}{2\alpha\beta}\,. \tag{2}$$

As a first application we derive a *parametric representation* of an oriented h-line $[u, v]$ with the h-distance σ, measured from one of its points and provided with a sign, as parameter.

For the h-lines $[u, \infty]$ and $[\infty, v]$ one has

$$\mathrm{x} = u + e^{\sigma} j\,, \quad \mathrm{x} = v + e^{-\sigma} j\,, \quad \sigma \in \mathbb{R}\,.$$

A proper h-line $[u, v]$ with $u, v \in \mathbb{C}$ is a semi-circle in U with centre $(u+v)/2$ and radius $|v-u|/2$ lying in a vertical half-plane. Let φ denote the angle, provided with a sign in the obvious manner, from the vertical radius to the radius ending at the point x. We have then the representation

$$\mathrm{x} = \frac{1}{2}(u + v + (v-u)\sin\varphi + |v-u|\, j\cos\varphi)\,, \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\,.$$

Since both σ and φ are preserved under horizontal translations and dilatations with centres in $\mathbb{C}$ and positive factors, σ can only depend on φ, and to determine the relation between σ and φ we may assume that $u = -1$ and $v = 1$. For the h-distance σ from j to $\sin\varphi + j\cos$ we have by (2)

$$\cosh\sigma = \frac{\sin^2\varphi + 1 + \cos^2\varphi}{2\cos\varphi} = \frac{1}{\cos\varphi}\,.$$

This gives $\sin^2\varphi = \tanh^2\sigma$, hence $\sin\varphi = \tanh\sigma$ since φ and σ have the same sign. The parametric representation required is therefore

$$\mathrm{x} = \frac{1}{2}(u + v + (v-u)\tanh\sigma + \frac{|v-u|}{\cosh\sigma} j)\,, \quad \sigma \in \mathbb{R}\,.$$

Applying the expression above for the h-distance, we obtain the following equation for the h-sphere with h-centre $c + \gamma j$ and h-radius ϱ:

$$|x-c|^2 + \xi^2 + \gamma^2 = 2\gamma\xi\cosh\varrho$$

or, equivalently,

$$|x-c|^2 + (\xi - \gamma\cosh\varrho)^2 = \gamma^2\sinh^2\varrho\,.$$

This shows that the h-sphere is a Euclidean sphere with centre $c + \gamma j\cosh\varrho$ and radius $\gamma\sinh\varrho$.

As a consequence we note that the topology in U induced by the h-metric is identical with the usual one.

The h-radii of an h-sphere are orthogonal to it. This is obvious for the two vertical ones. For the others it follows from the fact that the subgroup of $\mathscr{M}_3$ which leaves the h-centre fixed acts transitively on the h-sphere because $\mathscr{H}_3$ acts transitively on the set of orthogonal frames.

As an application hereof we show:

The h-distance of a point $c + \gamma j$, $\gamma > 0$, from the points of a proper h-line or an h-plane attains its minimum precisely at the foot of the perpendicular from $c + \gamma j$ to the h-line or h-plane. This minimum is called the h-distance of the point from the h-line or h-plane.

It may be assumed that the h-line is $[0, \infty]$. There exists precisely one h-sphere with h-centre $c + \gamma j$ which touches this h-line, namely that which is the Euclidean sphere with centre $c + (|c|^2 + \gamma^2)^{1/2} j$ of radius $|c|$. Its h-radius ϱ is determined by $\gamma \sinh \varrho = |c|$. The shortest connection between $c + \gamma j$ and the h-line is therefore the h-radius to the point of contact, and this h-radius is orthogonal to the h-line. This proves the statement for an h-line. For an h-plane it follows from the observation that the h-distance of $c + \gamma j$ from the foot of the perpendicular is minimal for every h-line through the foot in the h-plane.

Let P be an h-plane, and consider one of the h-half-spaces bounded by it. The set of points in this h-half-space which have a given positive h-distance δ from P is a spherical cap bounded by the horizon of P. It is called an *equidistance surface* of P, and P is called its *axial plane*.

To prove this, we may assume that P is the half-plane $\{x + \xi j \,|\, \operatorname{Im} x = 0\}$. The statement is then that the set in question is a half-plane also bounded by the real axis. Clearly, on every h-normal to P there is exactly one point which has h-distance δ from P and lies in the h-half-space considered. Let x be any such point. Its images under the translations parallel to the real axis and the dilatations with centre 0 and positive ratio make up the half-plane passing through x and bounded by the real axis. Since these transformations belong to $\mathscr{H}_3$ and map P onto itself, the statement follows.

Let P_1 and P_2 be ultraparallel planes. The h-distance of a point of P_1 from a point of P_2 attains its minimum precisely when the points are the intersections of P_1 and P_2 with their common normal. This minimum is called the h-distance of the h-planes.

To prove this, we may assume that P_1 is the vertical half-plane bounded by the real axis. Then P_2 is a hemisphere with centre in $\mathbb{C}$ and disjoint from P_1. There is exactly one equidistance surface of P_1, that is, a half-plane bounded by the real axis, which touches P_2. The minimal h-distance in question is the h-length of the h-perpendicular to P_1 from the point of contact. Being an arc of the semi-circle with centre on the real axis passing through the point of contact and lying in the plane orthogonal to P_1, the h-perpendicular is orthogonal to the equidistant surface and hence to P_2, which was to be shown.

As corollaries we obtain:

Let P be an h-plane and L an ultraparallel h-line. The h-distance of a point of P from a point of L attains its minimum precisely when the points are the intersections of P and L with their common h-normal. This minimum is called the h-distance of P and L.

Let L_1 and L_2 be non-parallel h-lines. The h-distance of a point of L_1 from a point of L_2 attains its minimum precisely when the points are the intersections of L_1 and L_2 with their common h-normal. This minimum is called the h-distance of L_1 and L_2.

Both statements are seen to be immediate consequences of the preceding one by taking the h-plane or h-planes through the h-line or h-lines orthogonal to the common normals into consideration.

By definition the h-distance of parallel h-lines and h-planes is 0. This is motivated by the fact that the infimum of the h-distance of a point of an h-plane or h-line from a point of a parallel h-plane or h-line is 0. To see this, assume that the common improper point is ∞ so that the h-planes and h-lines to be considered are vertical half-planes and half-lines. Let $x \neq \infty$ be a point on the horizon of one of the h-planes or the end in $\mathbb{C}$ of one of the h-lines, and let x' have the same meaning for a parallel h-plane or h-line. Then for all $\xi > 0$ the points $x + \xi j$ and $x' + \xi j$ lie on the respective h-planes or h-lines. Their Euclidean distance is independent of ξ, and therefore their h-distance tends to zero as ξ tends to infinity.

III.5 Transformation to the unit ball

Occasionally it is convenient to use the unit ball B of $\mathbb{J}_\infty$ instead of the upper half-space U as a model of hyperbolic space. As mentioned in II.5, there are elements of $\mathcal{M}_3$ which map U onto B. We shall use the anti-inversion in the sphere with centre $-j$ and radius $2^{1/2}$. It may be written

$$\mathrm{x}' = \Phi(\mathrm{x}) = -j - 2(\mathrm{x}+j)^{-1} = -j(\mathrm{x}-j)(\mathrm{x}+j)^{-1}.$$

With $\mathrm{x} = x + \xi j$ we have

$$|\mathrm{x}'|^2 = \frac{|x+(\xi-1)j|^2}{|x+(\xi+1)j|^2} = \frac{|x|^2+(\xi-1)^2}{|x|^2+(\xi+1)^2} < 1$$

if and only if $\xi > 0$, which shows that $\Phi(U) = B$. We note further that $\Phi(j) = 0$ and that Φ is involutory. The subgroup of $\mathcal{M}_3$ which maps B onto itself is $\Phi\mathcal{H}_3\Phi^{-1}$.

Observing that Φ is circle- and sphere-preserving and conformal, one easily sees how the notions of hyperbolic geometry defined in U are to be carried over to B. Here we shall be content with doing this for the Riemannian metric and the

h-distance. Letters standing for objects connected with B will be provided with a prime.

From

$$\mathrm{x} = -j - 2(\mathrm{x}' + j)^{-1}$$

we obtain

$$\begin{aligned} d\mathrm{x} &= 2(\mathrm{x}' + j)^{-1} d\mathrm{x}' (\mathrm{x}' + j)^{-1}, \\ d\mathrm{x}\, d\bar{\mathrm{x}} &= 4(\mathrm{x}' + j)^{-1} d\mathrm{x}' (\mathrm{x}' + j)^{-1} (\bar{\mathrm{x}}' - j)^{-1} d\bar{\mathrm{x}}' (\bar{\mathrm{x}}' - j)^{-1} \\ &= 4\, d\mathrm{x}'\, d\bar{\mathrm{x}}'\, |\mathrm{x}' + j|^{-4}. \end{aligned}$$

With $\mathrm{x} = x + \xi j$ and $\mathrm{x}' = x' + \xi' j$ we have

$$x + \xi j = -j - 2(\bar{x}' - (1 + \xi')j)\, |x' + (1 + \xi)j|^{-2},$$

hence
$$\begin{aligned} \xi &= -1 + 2(1 + \xi')\,(x'\bar{x}' + (1 + \xi')^2)^{-1} \\ &= (1 - x'\bar{x}' - \xi'^2)\,(x'\bar{x}' + (1 + \xi')^2)^{-1} \\ &= (1 - |\mathrm{x}'|^2)\, |\mathrm{x}' + j|^{-2}. \end{aligned}$$

For the Riemannian metric we thus obtain

$$\frac{d\mathrm{x}\, d\bar{\mathrm{x}}}{\xi^2} = \frac{4\, d\mathrm{x}'\, d\bar{\mathrm{x}}'}{(1 - |\mathrm{x}'|^2)^2}.$$

Let now $\mathrm{a} = a + \alpha j$ and $\mathrm{b} = b + \beta j$ be points in U with h-distance $\delta > 0$, further $\mathrm{a}' = a' + \alpha' j$ and $\mathrm{b}' = b' + \beta' j$ the corresponding points in B so that $\mathrm{a} = \Phi(\mathrm{a}')$ and $\mathrm{b} = \Phi(\mathrm{b}')$. In the preceding section we found

$$\sinh^2 \frac{\delta}{2} = \frac{|\mathrm{b} - \mathrm{a}|^2}{4\alpha\beta}.$$

To express this in terms of a' and b' we note that

$$\begin{aligned} |\mathrm{b} - \mathrm{a}| &= 2\, |(\mathrm{a}' + j)^{-1} - (\mathrm{b}' + j)^{-1}| \\ &= 2\, |(\mathrm{a}' + j)^{-1} (\mathrm{b}' - \mathrm{a}')\, (\mathrm{b}' + j)^{-1}| \\ &= 2\, \frac{|\mathrm{b}' - \mathrm{a}'|}{|\mathrm{a}' + j|\, |\mathrm{b}' + j|} \end{aligned}$$

and (see the expression for ξ above)

$$\alpha = \frac{1 - |\mathrm{a}'|^2}{|\mathrm{a}' + j|^2}, \quad \beta = \frac{1 - |\mathrm{b}'|^2}{|\mathrm{b}' + j|^2}.$$

Hence,

$$\sinh^2 \frac{\delta}{2} = \frac{|\mathrm{b}' - \mathrm{a}'|^2}{(1 - |\mathrm{a}'|^2)\,(1 - |\mathrm{b}'|^2)}.$$

This shows in particular that the h-sphere in B with h-centre 0 and h-radius ϱ, having the equation

$$|x'|^2 - \sinh^2 \frac{\varrho}{2}(1 - |x'|^2) = 0$$

which reduces to

$$|x'| = \tanh \frac{\varrho}{2},$$

is the Euclidean sphere with centre 0 and radius $\tanh \frac{\varrho}{2}$. We use this to prove:

An h-sphere S in U with h-radius ϱ provided with the h-metric is isometric to a Euclidean sphere with radius $\sinh \varrho$.

There is an element $f \in \mathscr{H}_3$ which carries the h-centre of S into j. Since $\Phi(j) = 0$, the image of S under $\Phi \circ f$ is the sphere S' with centre 0 and radius $\tanh \frac{\varrho}{2}$. This mapping being isometric in the h-sense it is sufficient to consider the metric on S'. Now $dx' d\bar{x}'$ restricted to S' is the usual spherical metric on a sphere of radius $\tanh \frac{\varrho}{2}$ and therefore $4\, dx' d\bar{x}' \left(1 - \tanh^2 \frac{\varrho}{2}\right)^{-2}$ the usual metric on a sphere of radius $2 \tanh \frac{\varrho}{2} \left(1 - \tanh^2 \frac{\varrho}{2}\right)^{-1} = \sinh \varrho$.

Finally we use Φ to show:

The subgroup of $\mathscr{H}_3$ which leaves a point $c \in U$ fixed is conjugate in $\mathscr{M}_3$ to the orthogonal group $O(3, \mathbb{R})$.

There is an element $f \in \mathscr{H}_3$ such that $f(c) = j$ and thus $\Phi \circ f(c) = 0$. By $\Phi \circ f$ the subgroup in question is transformed into the subgroup of $\mathscr{M}_3$ which leaves 0 fixed and maps the unit sphere onto itself. Then also the inverse, ∞, of 0 with respect to the unit sphere is fixed. The transformed subgroup consists therefore of the similarities leaving 0 and the unit sphere fixed, and consequently it is $O(3, \mathbb{R})$.

Notes to Chapter III

Hyperbolic geometry is here introduced as the geometry, with a suitable terminology, of the Möbius group of the upper half-space. This allows to derive many of the basic properties of hyperbolic notions using well-known facts of Euclidean geometry. A similar approach is due to Liebmann [13], 1. Aufl. Cf. also Coxeter [7].

Axiomatic treatments and the projective model are dealt with in many books. Of the newer ones we mention Busemann and Kelly [2], Perron [18], Coxeter [6], Baldus und Löbell [1], Lenz [12].

Many of the results of the following chapters have been known for a long time. For the older litterature the reader may consult the bibliography of Sommerville [27].

IV. The Isometry Group of Hyperbolic Space

IV.1 Characterization of the isometry group

The upper half-space U provided with the h-metric will be called *hyperbolic space*, and from now on the prefix h of terms denoting geometrical notions will be omitted. All these notions are to be understood in the sense of hyperbolic geometry. In case a notion is used in the Euclidean sense this will be indicated by a prefix e.

As already observed (cf. III.4) the elements of $\mathscr{H}_3$ are isometries of hyperbolic space. We shall show that there are no others.

Let ψ be an isometry. If a, b and c are distinct points which lie on a line in this order, their images satisfy

$$\delta(\psi(\mathrm{a}), \psi(\mathrm{b})) + \delta(\psi(\mathrm{b}), \psi(\mathrm{c})) = \delta(\psi(\mathrm{a}), \psi(\mathrm{c}))$$

and must therefore be collinear, with ψ(b) between ψ(a) and ψ(c). This implies that ψ must map oriented lines onto oriented lines.

Consider now two lines L and M which intersect orthogonally at a and a point b $\neq$ a on M. Since the distance of b from the points of L attains its minimum at a, the distance of ψ(b) from the points of $\psi(L)$ must attain its minimum at ψ(a). Consequently, $\psi(M)$ must intersect $\psi(L)$ orthogonally.

Let (L_1, L_2, L_3), where L_1, L_2, L_3 denote oriented lines, be an orthogonal frame (cf. III.2), and o be the intersection point of the lines. From the preceding remarks one infers that ψ maps (L_1, L_2, L_3) onto an orthogonal frame. There exists an element f of $\mathscr{H}_3$ such that

$$(f(L_1), f(L_2), f(L_3)) = (\psi(L_1), \psi(L_2), \psi(L_3)) .$$

Hence $f^{-1} \circ \psi$ leaves (L_1, L_2, L_3) pointwise fixed. It has to be shown that $f^{-1} \circ \psi$ is the identity.

Every line joining two distinct points known to be fixed under $f^{-1} \circ \psi$ must be pointwise fixed. This holds for every line which joins a point of L_1 and a point of L_2, both different from o. Since any line through o in the plane is pointwise fixed. The same argument applied to an arbitrary line $L \neq L_3$ through o with L_1 and L_2 replaced by L_3 and the intersection of the plane spanned by L_3 and L with the plane spanned by L_1 and L_2, respectively, yields that $f^{-1} \circ \psi$ is the identity in the entire space U, as claimed.

The argument above shows also that every isometry ψ of a plane H is the restriction to H of some element of $\mathscr{H}_3$. To see it, choose the frame (L_1, L_2, L_3)

such that L_1 and L_2 are contained in H. Then so are $\psi(L_1)$ and $\psi(L_2)$. There is therefore an $f \in \mathscr{H}_3$ such that $f(L_1) = \psi(L_1)$ and $f(L_2) = \psi(L_2)$ and which obviously maps H onto itself.

IV.2 Classification of the motions

The orientation-preserving isometries, that is, the elements of $\mathscr{H}_3^+$ restricted to U, will be called *motions*.

It is useful to take their actions on $\mathbb{C}_\infty$ into consideration. The restriction of $\mathscr{H}_3^+$ to $\mathbb{C}_\infty$, the Möbius group $\mathscr{M}_2$, has been shown to be the group of fractional linear transformations of $\mathbb{C}_\infty$. The properties of the latter group (cf. I.4) can be interpreted as properties of the group of motions.

Let $f \in \mathscr{H}_3^+$ be different from the identity. It has two fixed points u, v in $\mathbb{C}_\infty$ which may coincide. Assume first that they are distinct and consider the ordered pair (u, v). Then f has a definite multiplier

$$m = \not{R}(v, u, x, f(x)) \neq 0, 1, \infty, \quad x \in \mathbb{C} \setminus \{u, v\}.$$

The line $[u, v]$, but no other, is mapped onto itself with the orientation preserved. It is called the *axis* of the motion f. Since conjugate elements of $\mathscr{H}_3^+$ have conjugate restrictions to $\mathbb{C}_\infty$ and, hence, the same multiplier m, the geometrical significance of m can be inferred under the assumption that $[u, v] = [0, \infty]$. Then f is determined by the matrix $\begin{pmatrix} m^{1/2} & 0 \\ 0 & m^{-1/2} \end{pmatrix}$ with any choice of the square root (cf. I.4) and hence

$$f(x + \xi j) = mx + |m| \xi j.$$

In the Euclidean sense this is the dilatation with centre 0 and ratio $|m|$ composed with the rotation about the axis through the angle $\arg m$. The dilatation maps every half-plane bounded by the axis onto itself, and the image of an arbitrary point ξj, $\xi > 0$, of the axis is $|m| \xi j$. Now the (hyperbolic) distance from ξj to $|m| \xi j$, provided with a sign in accordance with the orientation of the axis, is $\log |m|$. Consequently the axis is translated into itself through this distance. Clearly, the rotation is also a rotation in the hyperbolic sense. The planes through the axis and the lines intersecting it at right angles are rotated about it through the angle $\arg m$. The positive sense of this rotation is that which together with the orientation of the axis gives the orientation of the space (cf. II.2).

For later use we observe that a plane which neither contains the axis nor is normal to it is not mapped onto itself by f.

With the motion f with oriented axis we associate the element

$$\delta(f) = \log m \neq 0$$

of the complex cylinder $\mathbb{A}$ (cf. I.2). We call it the *displacement* of f. Its real part is the distance through which the axis is translated and its imaginary part the angle through which the half-planes bounded by the axis are rotated. The motion f without an orientation of its axis determines δ only up to sign. If $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ determines f, one has (cf. I.4)

$$(1) \qquad 2\cosh\delta = m + 1/m = \operatorname{tr}\mathbf{f}^2 = \operatorname{tr}^2\mathbf{f} - 2\,.$$

The motions considered are classified as follows: f is called a *translation* if $\delta(f) \in \mathbb{R}\setminus\{0\}$, equivalently if $\operatorname{tr}\mathbf{f}$ is real and $\operatorname{tr}^2\mathbf{f} > 4$. It is called a *rotation* if $\delta(f) \neq 0$ is purely imaginary, equivalently if $\operatorname{tr}\mathbf{f}$ is real and $\operatorname{tr}^2\mathbf{f} < 4$. If $\operatorname{tr}\mathbf{f}$ is not real, f is called a *skrew motion.*

The rotation through the angle π about the axis $[u, v]$ is called the *half-turn* about the line $[u, v]$. Obviously it is involutory. The motion f determined by $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ is a half-turn if and only if $\delta(f) = \pi i$, equivalently, if and only if $\operatorname{tr}\mathbf{f} = 0$.

If a motion interchanges two distinct (proper or improper) points, it is a half-turn about a normal to the line joining the points.

If a motion f interchanges two proper points, it interchanges also the ends of the line joining them. We may therefore assume that $f(x) = y$ and $f(y) = x$ for some $x, y \in \mathbb{C}_\infty$, $x \neq y$. Since f has at least one improper fixed point u, we have $f(u) = u$. The three conditions determine f uniquely, and the half-turn about the normal to $[x, y]$ ending at u satisfies them.

As a consequence we note that the only involutory motions are the half-turns.

We consider now the motions with coinciding improper fixed points. They are all conjugate in $\mathscr{H}_3^+$ to

$$f_0(\mathrm{x}) = \mathrm{x} + 1$$

which has the fixed point ∞ (cf. I.4). Let $f = p \circ f_0 \circ p^{-1}$ be another such motion. Denote its fixed point $p(\infty)$ by u. The e-planes parallel to $\mathbb{C}$ are to be considered as e-spheres in $\operatorname{cl} U$ touching $\mathbb{C}_\infty$ at ∞. Their images under p are the e-spheres touching $\mathbb{C}_\infty$ at u. They are called the *horospheres* with (improper) *centre u.* The mutually parallel lines with common end u intersect these horospheres orthogonally. They are called the *diameters* of the latter. The planes with horizons passing through u, the *diametral planes* of the horospheres, intersect these in e-circles touching $\mathbb{C}_\infty$ at u, the *horocycles* with centre u. The motion f maps each of the horospheres with centre u onto itself, further, their diameters onto diameters, diametral planes onto diametral planes and horocycles onto horocycles. There is a pencil of mutually parallel diametral planes each of which is mapped onto itself, the *invariant planes* of f. No other plane is mapped onto itself. All this holds for f since it is obviously true for f_0.

The motions of the type considered are called *parallel motions* or *limit rotations*. If $f \in \mathscr{H}_3^+$ is determined by $\mathbf{f} \in \mathrm{SL}(3, \mathbb{C})$, it is a parallel motion if and only if $\mathbf{f} \neq \pm\mathbf{1}$ and $\operatorname{tr}^2\mathbf{f} = 4$. The relations above give the value $\delta(f) = 0$ for the dis-

placement. This is in accordance with the conventions concerning improper lines (cf. III.2). The fixed point u represents the *improper axis* of f, and the lines ending at u are its normals. By f such a normal is mapped onto a parallel line, and the distance of parallel lines is zero (cf. III.4).

The product of two distinct half-turns is a rotation, a parallel motion, a translation, or a skrew motion according as the axes of the half-turns intersect, are parallel, are ultraparallel, or are skew.

Every motion different from the identity is a product of two half-turns the axis of one of which may be chosen arbitrarily among the normals of the axis of the motion.

The first statement is obvious. In the case of parallel axes of the half-turns it is most easily obtained under the assumption that their common end is ∞. Then the half-turns are half-turns in the Euclidean sense about two vertical e-lines and their product an e-translation, thus, a parallel motion.

To prove the second statement, assume first that f is a motion with a proper axis $[u, v]$. Let h be a half-turn about a normal to $[u, v]$. Then $f \circ h$ interchanges u and v and is therefore a half-turn h' about a normal to $[u, v]$. Consequently $f = h' \circ h$. If f is a parallel motion, it may be assumed that $f(\mathrm{x}) = \mathrm{x} + 1$. The product of the half-turns about $[w, \infty]$ and $[w + \frac{1}{2}, \infty]$ for any choice of $w \in \mathbb{C}$ is f, as is easily seen by interpreting the transformations in the Euclidean sense.

Furthermore we observe:

Two motions different from the identity commute if and only if they have the same improper fixed points or both are half-turns with axes which intersect orthogonally.

Let f and g be such motions which commute. If $f(u) = u$ for some $u \in \mathbb{C}_\infty$, then $g \circ f \circ g^{-1} = f$ has the fixed point $g(u)$. This shows that each of f and g maps the set of fixed points of the other onto itself. Hence, if f has distinct improper fixed points u and v, either $g(u) = u, g(v) = v$ or $g(u) = v, g(v) = u$. In the second case g must be a half-turn with an axis intersecting $[u, v]$ orthogonally, and since f interchanges the improper fixed points of g, it must be the half-turn about $[u, v]$. If f is a parallel motion with fixed point u, we must have $g(u) = u$. Since another fixed point in $\mathbb{C}_\infty$ of g would also be a fixed point of f, it follows that g is likewise a parallel motion with fixed point u. Hence, the conditions in the statement are necessary.

The sufficiency is easily seen. That motions with a common proper axis commute follows for instance from the fact that they are determined by diagonal matrices when the axis is $[0, \infty]$. Parallel motions with a common fixed point u commute because they are Euclidean translations when $u = \infty$. For half-turns

the statement is an immediate consequence of the discussion above, which shows that the product of two half-turns about lines which intersect orthogonally is the half-turn about the common normal of these lines.

We observe that matrices $\mathbf{f}$ and $\mathbf{g}$ determining motions with the same improper fixed points commute, while matrices $\mathbf{f}$ and $\mathbf{g}$ determining half-turns with axes intersecting orthogonally anti-commute: $\mathbf{gf} = -\mathbf{fg}$. This is easily checked assuming that the axis of one of the half-turns is $[0, \infty]$.

IV.3 Reversals

We consider now the elements of $\mathscr{H}_3^-$, that is, the orientation-reversing isometries. They will be called *reversals*. As shown in II.5, a reversal f^* is determined by a matrix of the form $\mathbf{f}j$, where $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$. If f denotes the motion determined by $\mathbf{f}$, we have

$$f^*(x + \xi j) = f(-j(x + \xi j)j) = f(\bar{x} + \xi j)\,.$$

Let g be a motion determined by the matrix $\mathbf{g}$. Then $g \circ f^*$ is determined by $\mathbf{g}\mathbf{f}j$, while $f^* \circ g$ is determined by $\mathbf{f}j\mathbf{g} = \mathbf{f}\bar{\mathbf{g}}j$. If g^* is the reversal determined by $\mathbf{g}j$, the motion $g^* \circ f^*$ is determined by

$$\text{(1)} \qquad \mathbf{g}j\mathbf{f}j = -\mathbf{g}\bar{\mathbf{f}}\,,$$

in particular f^{*2} by $-\mathbf{f}\bar{\mathbf{f}}$. The inverse f^{*-1} of f^* is determined by

$$\text{(2)} \qquad -j\mathbf{f}^{-1} = -\bar{\mathbf{f}}^{-1}j$$

since $-j\mathbf{f}^{-1}\mathbf{f}j = \mathbf{1}$ and $\mathbf{f}j(-j\mathbf{f}^{-1}) = \mathbf{1}$. (It should be observed that f^{-1*} is determined by $\mathbf{f}^{-1}j$ and, thus, in general different from f^{*-1}).

For the conjugate $p^* \circ f \circ p^{*-1}$ of a motion f by a reversal p^* we have with obvious notations

$$\text{(3)} \qquad -\mathbf{p}j\mathbf{f}j\mathbf{p}^{-1} = \mathbf{p}\bar{\mathbf{f}}\mathbf{p}^{-1}\,,$$

for the conjugate $p \circ f^* \circ p^{-1}$ of a reversal f^* by a motion p

$$\text{(4)} \qquad \mathbf{p}\mathbf{f}j\mathbf{p}^{-1} = \mathbf{p}\mathbf{f}\bar{\mathbf{p}}^{-1}j$$

and for the conjugate $p^* \circ f^* \circ p^{*-1}$ of a reversal f^* by a reversal p^*

$$\text{(5)} \qquad -\mathbf{p}j\mathbf{f}j^2\mathbf{p}^{-1} = \mathbf{p}\bar{\mathbf{f}}\bar{\mathbf{p}}^{-1}j\,.$$

Let $a, b, c, d \in \mathbb{C}_\infty$ such that not all of them coincide, and let f^* be a reversal. Then

$$\mathcal{R}(f^*(a), f^*(b), f^*(c), f^*(d)) = \overline{\mathcal{R}(a, b, c, d)}\,.$$

To see this, observe that f^* is composed of a motion and the reversal $x + \xi j \mapsto -j(x + \xi j)j = \bar{x} + \xi j$, and that the restriction of the latter to $\mathbb{C}_\infty$ is $x \mapsto \bar{x}$. The left hand side is therefore equal to $\not{R}(\bar{a}, \bar{b}, \bar{c}, \bar{d})$.

The multiplier $m(f) = \not{R}(v, u, x, f(x))$ of a motion f with the improper fixed points u, v is changed to its complex conjugate value if f is transformed to $p^* \circ f \circ p^{*-1}$ by a reversal p^*. Again it is sufficient to consider the case where the restriction of p^* to $\mathbb{C}_\infty$ is $x \mapsto \bar{x}$. Then $p^* \circ f \circ p^{*-1}(x) = \overline{f(\bar{x})}$, and hence

$$\begin{aligned} m(p^* \circ f \circ p^{*-1}) &= \not{R}(\bar{v}, \bar{u}, x, \overline{f(\bar{x})}) \\ &= \overline{\not{R}(v, u, \bar{x}, f(\bar{x}))} = \overline{m(f)}, \end{aligned}$$

as claimed. For the displacement of the motion f we obviously also have

$$\delta(p^* \circ f \circ p^{*-1}) = \overline{\delta(f)}.$$

There is no analogue of the multiplier for a reversal f^*, even if it has two improper fixed points u, v, because $\not{R}(v, u, x, f^*(x))$ is not independent of x. If $\mathbf{f}j$ is a matrix determining f^*, the trace of $\mathbf{f}$ is not invariant. However, the trace of the matrix $-\mathbf{f}\bar{\mathbf{f}}$ determining f^{*2} is invariant under motions and reversals. Indeed, if $\mathbf{p}$ is a matrix determining a motion p, a matrix determining $(p \circ f^* \circ p^{-1})^2$ is

$$\mathbf{p}\mathbf{f}\bar{\mathbf{p}}^{-1} j \mathbf{p}\mathbf{f}\bar{\mathbf{p}}^{-1} j = -\mathbf{p}\mathbf{f}\bar{\mathbf{f}}\mathbf{p}^{-1},$$

and if $\mathbf{p}j$ determines the reversal p^*, a matrix determining $(p^* \circ f^* \circ p^{*-1})^2$ is

$$\mathbf{p}\bar{\mathbf{f}}\bar{\mathbf{p}}^{-1} j \mathbf{p}\bar{\mathbf{f}}\bar{\mathbf{p}}^{-1} j = -\mathbf{p}\bar{\mathbf{f}}\mathbf{f}\mathbf{p}^{-1}.$$

It will be convenient to consider $\operatorname{tr}(\mathbf{f}\bar{\mathbf{f}}) = \operatorname{tr}(\bar{\mathbf{f}}\mathbf{f})$. Clearly, this is a real number, actually

$$\operatorname{tr}(\mathbf{f}\bar{\mathbf{f}}) \geqq -2.$$

Indeed, with the usual notations, observing that

$$f_{11} f_{22} = f_{12} f_{21} + 1,$$

we have

$$\begin{aligned} \operatorname{tr}(\mathbf{f}\bar{\mathbf{f}}) &= |f_{11}|^2 + |f_{22}|^2 + f_{12}\bar{f}_{21} + \bar{f}_{12} f_{21} \\ &\geqq 2\,|f_{11}|\,|f_{22}| + f_{12}\bar{f}_{21} + \bar{f}_{12} f_{21} \\ &= 2\,|f_{12} f_{21} + 1| + f_{12}\bar{f}_{21} + \bar{f}_{12} f_{21} \\ &\geqq 2\,|f_{12} f_{21}| - 2 + f_{12}\bar{f}_{21} + \bar{f}_{12} f_{21} \geqq -2. \end{aligned}$$

This implies that the square of a reversal cannot be a skrew motion. The geometrical significance of $\operatorname{tr}(\mathbf{f}\bar{\mathbf{f}})$ will be discussed later.

We consider now the involutory reversals. Clearly, a necessary and sufficient

condition for a reversal f^* determined by the matrix $\mathbf{f}j$ being involutory is $\mathbf{f}\bar{\mathbf{f}} = \pm 1$.

One type of involutory reversals are the *plane-reflections*. Given a plane P, the reflexion f^* with the *mirror* P is defined in the obvious way: $f^*(x)$ is that point of the normal to P through x which has the same distance from P as x and is separated from x by P unless $x \in P$. That f^* belongs to $\mathscr{H}_3^-$, follows for instance from the fact that, in the Euclidean sense, it is the inversion in the e-sphere containing P. Since every reversal conjugate to a plane-reflection is also a plane-reflection and $\mathscr{H}_3$ acts transitively on the set of planes, the plane-reflections form a conjugacy class. Since $x + \xi j \mapsto \bar{x} + \xi j$, determined by $\mathbf{1}j$, is a plane-reflection, we have $\mathbf{f}\bar{\mathbf{f}} = \mathbf{1}$ for all plane-reflections.

A second type of involutory reversals are the *point-reflections*. Given a proper point c, the reflection f^* with the *centre* c is defined in the obvious way: $f^*(c) = c$ and, for a proper point $x \neq c$, $f^*(c)$ is that point of the line joining x and c which has the same distance from c as x and is separated by c from x. Since f^* is the product of a half-turn about a line through c and the reflection in the plane through c and orthogonal to that line, it is an element of $\mathscr{H}_3^-$. The product of any plane-reflection and a half-turn about a normal to its mirror is the point-reflection in the intersection, and the two transformations commute. Obviously, the set of point-reflections is a conjugacy class of $\mathscr{H}_3$. Now $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ determines the half-turn about the line $[-i, i]$ which intersects the plane with the real axis as horizon orthogonally at the point j. Hence $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} j$ determines the point-reflection with centre j. Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -\mathbf{1}$, we have $\mathbf{f}\bar{\mathbf{f}} = -\mathbf{1}$ for all point-reflections.

There are no other involutory reversals. Assume that $f^* \in \mathscr{H}_3^-$ is involutory. If it has an improper fixed point u, there is some $v \in \mathbb{C}$ different from u such that $f^*(v) = v' \neq v$ and, of course, $f^*(v') = v$. Since these relations together with $f^*(u) = u$ determine f^* uniquely (cf. II.5) and the reflection in the plane whose horizon contains u and which is orthogonal to $[v, v']$ satisfies them, f^* must be this plane-reflection. If f^* has no improper fixed points, consider any two points $v, w \in \mathbb{C}_\infty$ such that $w \neq v$, $w \neq v' = f^*(v)$. Then we have $f^*(v) = v'$, $f^*(v') = v$, $f^*(w) = w'$, $f^*(w') = w$. These relations determine f^* uniquely. Now the lines $[v, v']$ and $[w, w']$, each of which is mapped onto itself, must intersect. Otherwise the points at which they intersect their common normal would be fixed and hence the latter point-wise fixed, contrary to the assumption that f^* has no improper fixed points. Since the reflection in the point at which $[v, v']$ and $[w, w']$ intersect satisfies the relations, f^* must be this point-reflection.

Before discussing the non-involutory reversals we observe:

The product of two plane-reflections with ultraparallel mirrors is a translation

with the common normal of these as axis. The displacement is twice the distance of the mirrors. Clearly, every translation can be obtained in this way, where one of the mirrors may be chosen arbitrarily among the normal planes of the axis.

The product of two-reflections with parallel mirrors is a parallel motion which carries each of the planes orthogonal to both mirrors into itself. Every parallel motion can be obtained in this way, where one of the mirrors may be chosen arbitrarily among the planes which intersect the planes which are mapped onto themselves orthogonally. All this is most easily seen assuming that the common point of the horizons of the mirrors, thus the fixed point of the parallel motion, is ∞.

The product of two plane-reflections with intersecting mirrors is a rotation with the line of intersection as axis through twice the angle between the mirrors. Every rotation can be obtained in this way, where one of the mirrors may be chosen arbitrarily among the planes containing the axis.

The product of two distinct point-reflections is a translation with the line joining the centres as axis through twice the distance of the centres. Every translation can be obtained in this way, where one of the centres may be chosen arbitrarily on the axis.

The product of a point-reflection and a plane-reflection is the half-turn about the normal to the mirror through the centre if the centre lies in the mirror, otherwise a skrew motion through twice the distance of the centre from the mirror, through the angle π, *and with the normal to the mirror through the centre as axis. Every such motion can be obtained in this way. Either the centre on the axis or the mirror normal to the axis may be chosen arbitarily.*

Every reversal is the product of a half-turn and a plane-reflection.

As can be inferred from previous remarks, a plane-reflection f^* is the product of a plane-reflection with a mirror orthogonal to the mirror of f^* and the half-turn about the intersection of the mirrors, these two transformations taken in any order. Further, a point-reflection f^* is the product of a plane-reflection with a mirror passing through the centre of f^* and the half-turn about the normal to the mirror through this centre, these two transformations taken in any order.

Let now f^* be a non-involutory reversal. Then f^{*2} is a motion with two, possibly coinciding, fixed points u and v. Consider a point $x \in \mathbb{C}$ different from u and v. The three points $x, y = f^{*-1}(x), z = f^*(x)$ are then distinct; for otherwise f^{*2} would have a fixed point different from u and v. Denote by r^* the reflection in the plane normal to the line $[y, z]$ whose horizon contains x (cf. III.2). Then $f^* \circ r^*(x) = f^*(x) = z$ and $f^* \circ r^*(z) = f^*(y) = x$. Consequently, $f^* \circ r^*$ is a half-turn h and thus $f^* = h \circ r^*$, as claimed.

Every reversal is the product $t^* \circ s^* \circ r^*$ of three plane-reflections t^*, s^*, and *r^* such that the mirror of t^* intersects those of s^* and r^* orthogonally so that t^* commutes with the others.*

Let $f^* = h \circ r^*$, where h is a half-turn. Now h may be replaced by the product $t^* \circ s^*$ of two plane-reflections with mirrors intersecting orthogonally in the axis of h. The mirror of t^* may be chosen orthogonally to that of r^*. Indeed, there is exactly one plane through the axis of h and orthogonal to the mirror of r^*, unless the axis is orthogonal to that mirror when any plane through the axis satisfies the requirement (cf. III.2). This proves the statement.

We use this result to classify the reversals f^* according to the mutual position of the mirrors of r^* and s^*, omitting the case where these coincide and f^* is the plane-reflection t^*.

If the mirrors of r^* and s^* are ultraparallel, the mirror of t^* passes through their common normal $[u, v]$. Hence, f^* is the translation $s^* \circ r^*$ preceded or followed by the reflection t^* in a plane through the axis $[u, v]$. The mirror of t^* is uniquely determined by f^* since it is the only plane through $[u, v]$ such that each of its half-planes bounded by $[u, v]$ is mapped onto itself by f^*. A reversal f^* of this type is called a *glide reflection* with axis $[u, v]$. The mirror of t^* is also called the *mirror* of f^*. The motion f^{*2} is a translation. Since there is a motion which carries the axis $[u, v]$ into $[0, \infty]$ and the mirror of f^* into the plane $\{x + \xi j \mid \operatorname{Im} x = 0\}$, a glide reflection f^* which translates the axis $[u, v]$ through the distance $\delta > 0$ is conjugate to that determined by the matrix $\begin{pmatrix} e^{\delta/2} & 0 \\ 0 & e^{-\delta/2} \end{pmatrix} j$, that is, to $x + \xi j \mapsto e^{\delta}(\bar{x} + \xi j)$. It follows that

$$\text{(6)} \qquad \operatorname{tr}(\mathbf{f}\bar{\mathbf{f}}) = 2 \cosh \delta > 2\,,$$

$\mathbf{f}j$ with $\det \mathbf{f} - 1$ denoting a matrix determining f^*.

If the mirrors of r^* and s^* are parallel, the horizon of the mirror of t^* passes through their common improper point u. In this case f^* is the parallel motion $s^* \circ r^*$ preceded or followed by the reflection t^* in an invariant plane of the parallel motion. This plane, the *mirror* of f^*, is uniquely determined by f^* because it is the only one of the invariant planes of $s^* \circ r^*$ which is also invariant under f^*. A reversal f^* of this type is called a *parallel reflection*. Its square is a parallel motion, and u is its only fixed point which may be considered as its improper axis. Since there is a motion carrying the mirrors of r^*, s^*, and t^* into the planes

$$\{x + \xi j \mid \operatorname{Re} x = 0\}\,, \quad \{x + \xi j \mid \operatorname{Re} x = \tfrac{1}{2}\}\,, \quad \{x + \xi j \mid \operatorname{Im} x = 0\}\,,$$

respectively, every parallel reflection f^* is conjugate to $x + \xi j \mapsto \bar{x} + \xi j + 1$. As this is determined by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} j$, one has, with usual notation,

$$(7) \qquad \operatorname{tr}(\mathbf{f}\bar{\mathbf{f}}) = 2$$

for any parallel reflection f^*.

If the mirrors of r^* and s^* intersect, the mirror of t^* is normal to their common line $[u, v]$. In this case f^* is the rotation $s^* \circ r^*$ about $[u, v]$ preceded or followed by the reflection t^* in a plane normal to the axis $[u, v]$. This plane, the *mirror* of f^*, being the only one normal to $[u, v]$ which is mapped onto itself by f^*, is uniquely determined by f^*. A reversal f^* of this type is called a *rotary reflection*. It interchanges u and v, it has exactly one fixed point c, the intersection of $[u, v]$ and the mirror of t^*, and its square is a rotation about $[u, v]$. There is a motion which carries $[u, v]$ into $[0, \infty]$ and the mirror of t^* into the plane the horizon of which is the unit circle in $\mathbb{C}_\infty$. The reflection in this plane is $x + \xi j \mapsto -i(\bar{x} + \xi j)^{-1}\, i$, determined by the matrix $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} j$. Indeed the restriction $x \mapsto 1/\bar{x}$ to $\mathbb{C}_\infty$ leaves every point of the unit circle fixed. The rotation about $[0, \infty]$ through the angle φ, $0 < \varphi \leqq \pi$, is determined by the matrix $\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$. The rotary reflection composed of these two transformations is determined by $\begin{pmatrix} 0 & ie^{i\varphi/2} \\ ie^{-i\varphi/2} & 0 \end{pmatrix} j$ and is therefore

$$x + \xi j \mapsto -ie^{i\varphi/2}(\bar{x} + \xi j)^{-1}\, ie^{i\varphi/2} = (e^{i\varphi}x + \xi j)\,(x\bar{x} + \xi^2)^{-1}\,.$$

It follows that every rotary reflection f^* such that $s^* \circ r^*$ rotates through the angle φ is conjugate to this special one. Since

$$\begin{pmatrix} 0 & ie^{i\varphi/2} \\ ie^{-i\varphi/2} & 0 \end{pmatrix} \begin{pmatrix} 0 & -ie^{-i\varphi/2} \\ -ie^{i\varphi/2} & 0 \end{pmatrix} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix},$$

we have for $\mathbf{f}j$ determining f^*

$$(8) \qquad -2 \leqq \operatorname{tr}(\mathbf{f}\bar{\mathbf{f}}) = 2\cos\varphi < 2\,.$$

Point-reflections are rotary reflections with $\varphi = \pi$ and therefore characterized by $\operatorname{tr}(\mathbf{f}\bar{\mathbf{f}}) = -2$, in accordance with $\mathbf{f}\bar{\mathbf{f}} = -1$. A reversal f^* is a point-reflection if and only if it is the product of three plane-reflections whose mirrors have a proper point in common and are pairwise orthogonal. They may be taken in any order.

As a consequence of the preceding classification we note that two reversals f^* and g^* are conjugate if and only if both are plane-reflections or none of them is, and

$$\operatorname{tr}(\mathbf{g}\bar{\mathbf{g}}) = \operatorname{tr}(\mathbf{f}\bar{\mathbf{f}})$$

for determining matrices $\mathbf{f}j$ and $\mathbf{g}j$.

IV.4 The isometry group of a plane

We consider a plane H in the hyperbolic space U. As stated in IV.1, every isometry of H is the restriction to H of an isometry of U which maps H onto itself. Every isometry of H is the restriction of precisely two elements of $\mathscr{H}_3$ any one of which is obtained from the other by composition with the reflection in H. This implies that the isometry group $\mathscr{H}_2$ of H can be obtained by restricting to H either the subgroup of $\mathscr{H}_3^+$ which leaves H invariant or the subgroup of $\mathscr{H}_3$ under which each of the half-spaces bounded by H is invariant. It is slightly more convenient to consider the first possibility.

In order to obtain a simple representation of $\mathscr{H}_2$ we may assume that $H = \{x + \xi j \mid \operatorname{Im} x = 0\}$. Clearly, the restrictions to H of the elements f of $\mathscr{H}_3^+$ determined by real matrices $\mathbf{f} \in \mathrm{SL}(2, \mathbb{R})$ are isometries of H. They are even motions in the sense that they preserve orientation in H. This can, for instance, be inferred from the fact that these transformations form a group, isomorphic to $\mathrm{SL}(2, \mathbb{R})/\{\mathbf{1}, -\mathbf{1}\}$, which is generated by $x + \xi j \mapsto x + \xi j + q$, $q \in \mathbb{R}$, and $x + \xi j \mapsto -(x + \xi j)^{-1}$ (cf. I.4). The latter maps restricted to $x \in \mathbb{R}_\infty$ obviously preserve orientation in H. That every motion in H is determined by a real matrix $\mathbf{f}$ with $\det \mathbf{f} = 1$ can be seen as follows. Let f be such a motion, and let $f^{-1}(\infty) = a$, $f^{-1}(0) = b$ and $f^{-1}(1) = c$. Then $a, b, c \in \mathbb{R}_\infty$ and they are distinct. In the cyclical order a, b, c they determine the same orientation of $\mathbb{R}_\infty$ as $\infty, 0, 1$. For $x \in \mathbb{R}_\infty$ we have (cf. I.4)

$$f(x) = R(a, b, c, x) = \frac{(c-a)\,(x-b)}{(c-b)\,(x-a)}$$

with obvious modifications if one of a, b, c is ∞.

This shows that f is determined by a real matrix with determinant $(c-a)\,(c-b)\,(b-a) > 0$ and hence by a matrix $\mathbf{f} \in \mathrm{SL}(2, \mathbb{R})$. Thus, $\mathscr{H}_2^+$ is isomorphic to $\mathrm{SL}(2, \mathbb{R})/\{\mathbf{1}, -\mathbf{1}\}$.

The whole group $\mathscr{H}_2$ is obtained by adjoining the restriction to H of the half-turn about the j-axis, that is, the reflection in this line. The half-turn, being determined by the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, is $x + \xi j \mapsto -x + \xi j$. Hence we have:

The elements of $\mathscr{H}_2^+$ are the transformations

$$x + \xi j \mapsto f(x + \xi j)\,, \quad x \in \mathbb{R}\,,$$

those of $\mathscr{H}_2^-$

$$x + \xi j \mapsto f(-x + \xi j)\,, \quad x \in \mathbb{R}\,,$$

where the transformations f are determined by the matrices $\mathbf{f} \in \mathrm{SL}(2, \mathbb{R})$.

Observing that the quaternions $x + \xi j$ with $x, \xi \in \mathbb{R}$ form a subfield of the division ring $\mathbb{H}$ which is isomorphic to $\mathbb{C}$, we may here replace j by the imaginary

unit i. This amounts to the replacement of H by the upper half-plane of $\mathbb{C}$. Writing z instead of $x + \xi i$, we obtain the familiar representation

$$z \mapsto f(z)$$

for motions and

$$z \mapsto f(-\bar{z})$$

for the reversals of the hyperbolic plane.

In order to classify the isometries let us consider again the plane $H = \{x + \xi j \mid \operatorname{Im} x = 0\}$. Since the motions then are determined by real matrices $\mathbf{f}$ and hence $\operatorname{tr} \mathbf{f} \in \mathbb{R}$, the corresponding elements of $\mathscr{H}_3^+$ are *translations, rotations,* and *parallel motions* (cf. IV.2). For several reasons it is obvious that the axis of a translation must be contained in H, the axis of a rotation must be orthogonal to H, and the improper axis of a parallel motion must be contained in $\mathbb{R}_\infty$. The reversals of H are the restrictions to H of the elements of $\mathscr{H}_3^+$ which interchange the half-spaces bounded by H. Clearly, the half-turns about lines in H have this property. Their restrictions are the *line-reflections* in H. The only other motions leaving H invariant and interchanging the half-spaces are the skrew motions with axis in H and rotation angle π. The restriction of such a skrew motion is composed of a translation and the reflection in its axis and is thus a *glide reflection* in H.

The involutory elements of $\mathscr{H}_2^+$ are the rotations through π, the *half-turns,* about a point of H. Those of $\mathscr{H}_2^-$ are the line-reflections. Representation of the elements of $\mathscr{H}_2$ as products of involutory ones are obtained immediately by specializing the results concerning the representation of the elements of $\mathscr{H}_3^+$ as products of two half-turns about lines (cf. IV.2).

A translation in H is the product of the reflections in two ultraparallel lines in H. Its axis is their common normal, and one of them may be chosen arbitrarily among the normals of the axis. A translation is also the product of the half-turns about two points of its axis, and one of these may be chosen arbitrarily on the axis.

A rotation is the product of the reflections in two lines passing through its centre, one of which may be chosen arbitrarily among such lines. A half-turn is obtained if the lines are orthogonal. In this case the two reflections commute.

A parallel motion is the product of reflections in two parallel lines, whose common end is the improper axis of the motion. One of the lines may be chosen arbitrarily among the lines with this end.

A glide reflection is the product of a half-turn and a line-reflection and also the product of a line-reflection and a half-turn. Its axis is that normal to the reflection line which passes through the centre of the half-turn. One may chose arbitrarily either the centre on the axis or the reflection line among the normals to the axis. Since the half-turn may be replaced by the product of the reflections in the axis and the reflection in the normal of the latter through the centre we also have: A

glide reflection is the product of three line-reflections, namely in two ultraparallel lines and in their common normal, its axis. The last named reflection commutes with the other two.

IV.5 The spherical and cylindrical surfaces

We consider now the surfaces in hyperbolic space which are e-spheres or parts of e-spheres (including the e-planes) more closely. Obviously, there are three cases. An e-sphere S may be contained entirely in U, it may be contained in $cl\,U$ and touch $\mathbb{C}_\infty$, or it may intersect $\mathbb{C}_\infty$. If S has e-centre $c_e + \gamma_e j$ and e-radius $r > 0$, we have the first, second, or third case according as $r < \gamma_e$, $r = \gamma_e$, or $r > |\gamma_e|$. If S is an e-plane, we have the second or the third case according as S is e-parallel to $\mathbb{C}_\infty$ and contained in U or intersects $\mathbb{C}_\infty$.

From results in III.4 it follows that S in the first case is a *sphere* with centre $c + \gamma j$ and radius $\varrho > 0$ where $c = c_e$ and ϱ and $\gamma > 0$ are determined by $\gamma \sinh \varrho = r$, $\gamma \cosh \varrho = \gamma_e$, hence

$$(1) \qquad \tanh \varrho = \frac{r}{\gamma_e}, \quad \gamma^2 = \gamma_e^2 - r^2 .$$

In III.5 it was shown that S with the metric induced by the hyperbolic metric is isometric with an e-sphere of e-radius $\sinh \varrho$. The role of the e-great circles is taken over by the circles in which S is intersected by the diametral planes, that is, the planes through the centre $c + \gamma j$. Further it was shown that the subgroup of $\mathcal{H}_3$ which leaves S invariant is isomorphic to the orthogonal group $\mathrm{O}(3, \mathbb{R})$.

Through every point x of a sphere S there is a plane orthogonal to the radius ending at x, the *tangent plane* of S at x. All its points except x, are in the exterior of S. The *tangents* of S at x are the lines in the tangent plane which pass through x. Through any lines disjoint with S, that is, contained in the exterior of S, there pass exactly two tangent planes to S. Since the line may be assumed to be a vertical e-half-line, this is a consequence of its Euclidean analogue.

We consider now the e-spheres in $cl\,U$ which touch $\mathbb{C}_\infty$ (including the e-planes parallel to $\mathbb{C}_\infty$). These are the *horospheres* defined in IV.2. Clearly, the elements of $\mathcal{H}_3$ map horospheres onto horospheres. Any two horospheres are congruent in the sense that there are elements of $\mathcal{H}_3$, even of $\mathcal{H}_3^+$, which map one onto the other. To see this consider any horosphere S and let $u \in \mathbb{C}_\infty$ be its (improper) centre. If $p \in \mathcal{H}_3^+$ is such that $p(u) = \infty$, then $p(S)$ is an e-plane parallel to $\mathbb{C}_\infty$. A suitable translation with axis $[0, \infty]$, that is, a Euclidean dilatation with centre 0, maps $p(S)$ onto the horosphere S_0, the e-plane parallel to $\mathbb{C}_\infty$ and passing through the point j. This implies the statement and shows that the following statements concerning horospheres only have to be checked for $u = \infty$ and S_0, in which case they are obvious.

Let S be a horosphere. With the metric induced on it by the hyperbolic metric it is isometric with the Euclidean plane. The role of the lines is taken over by the *horocycles* on S, that is, the intersections of S with its diametral planes. The subgroup of $\mathscr{H}_3$, whose elements map S onto itself, is isomorphic with the isometry group of the Euclidean plane. It consists of the parallel motions leaving the centre u of S fixed, the rotations about axes ending at u, reflections in diametral planes, and parallel reflections leaving u fixed. The subgroup of $\mathscr{H}_3$ of all elements leaving u fixed contains in addition all translations, skrew motions, and glide reflections with axes ending at u. It is isomorphic with the group of similarities of the Euclidean plane. Its elements map each horosphere with centre u onto a concentric one.

A horosphere S divides the space U into two domains. One of these has only one improper point, the centre u of S, on its boundary. This domain is called the *interior* and the other one the *exterior* of S. In particular, if S is a horizontal e-plane, the interior is the e-half-space above S. It is often convenient to orient the diameters from the interior to the exterior and to provide the distance of a point from S, measured on the diameter through the point, with a sign according to this orientation. The diameters of S are orthogonal to S. The *tangent planes* and *tangents* of S are defined as for spheres. With the exception of the point of contact, they lie in the exterior of S. Through every line in the exterior there pass precisely two tangent planes. Assuming the line to be $[0, \infty]$ this is again a consequence of the Euclidean analogue. We note further that a line which has points in common with the interior of S intersects S in two proper points unless it is a diameter.

The distance between two concentric horospheres S and S' measured on a diameter is independent of the choice of the latter. For the ordered pair (S, S') this distance δ is provided with a sign in accordance with the orientation of the diameters. Hence it is positive or negative according as S' lies in the exterior or interior of S. The bijection of S onto S' which to $\mathrm{x} = x + \xi j \in S$ lets correspond the intersection $\mathrm{x}' = x' + \xi' j$ of S' and the diameter through x is a "similarity" in the sense that the metrics induced on S and S' satisfy

$$\frac{d\mathrm{x}'\, d\bar{\mathrm{x}}'}{\xi'^2} = e^{2\delta} \frac{d\mathrm{x}\, d\bar{\mathrm{x}}}{\xi^2}.$$

This is evident since S and S' may be assumed to be horizontal e-planes. If S is chosen as the one passing through j, then S' passes through $e^{-\delta} j$, and thus $x' = x$, $\xi = 1$, and $\xi' = e^{-\delta}$.

For later use we mention that the relation between the length s of an arc of a horocycle and the length σ of its chord is

$$(2) \qquad s = 2 \sinh \frac{\sigma}{2}.$$

Since it may be assumed that the horocycle is the e-line through j parallel to the real axis and that the endpoints of the arc are $-\frac{s}{2}+j$ and $\frac{s}{2}+j$, this follows from the expression for the distance of two points derived in III.4.

We consider now the third of the cases mentioned in the beginning of this section. The cap contained in U of an e-sphere (or e-plane) intersecting $\mathbb{C}_\infty$ is a plane if orthogonal to $\mathbb{C}_\infty$ otherwise an *equidistant surface* as defined in III.4. The intersection of the e-sphere containing an equidistant surface S with $\mathbb{C}_\infty$ will be called the *horizon* of S. The plane with the same horizon, the *axial plane* of S, will be denoted by A_S. An equidistant surface S divides the space U into two domains. The one which contains A_S is called the *interior*, the other one the *exterior* of S. The normals of A_S, called the *diameters* of S, intersect S also orthogonally. This is obvious since it may be assumed that A_S is a vertical e-half-plane, so S is an e-half-plane bounded by the same e-line in $\mathbb{C}$. The planes orthogonal to A_S are called the *diametral planes* of S.

An equidistant surface S has a *tangent plane* at each of its points, namely the plane normal to the diameter through that point. Apart from the point of contact the tangent plane is contained in the exterior of S. Through every line in the exterior of S there pass exactly two tangent planes. To see it, suppose the line to be $[0, \infty]$. Then the e-sphere containing S has its e-centre in the upper half-space U since S separates the line from the axial plane A_S. Hence, the statement follows also in this case from the Euclidean analogue. The *tangents* of S are defined in the obvious manner.

The *distance* δ from the axial plane A_S to the equidistant surface S measured on a diameter is, by definition, independent of the choice of the latter. Occasionally it is convenient to provide the distance δ with a sign by orienting A_S and letting δ be positive or negative according as S lies on the positive or negative side of A_S. There is a simple relation between the angle φ which A_S and S form at the common horizon and the distance δ, namely:

$$(3) \qquad \cosh\delta\cos\varphi = 1$$

or, equivalently,

$$(4) \qquad \tanh\delta = \sin\varphi\,, \quad \sinh\delta = \tan\varphi\,,$$

valid also if δ and φ are provided with signs in the obvious manner. For the proof assume that the horizon of S and A_S is the real axis of $\mathbb{C}$, so S and A_S are e-half-planes bounded by it, A_S vertical. The diameter through j in A_S is contained in the unit e-circle in the e-plane spanned by the imaginary axis and the j-axis. It intersects S at the point

$$i\sin\varphi + \cos\varphi j\,.$$

For its distance δ from j we have (cf. III.4)

$$\cosh\delta = \frac{\sin^2\varphi + 1 + \cos^2\varphi}{2\cos\varphi},$$

which proves the statement.

To determine the metric induced on S by the hyperbolic metric in U we consider the bijection Φ of A_S onto S in which points on the same diameter correspond to each other. If we again assume that the horizon of A_S and S is the real axis, Φ is the restriction to A_S of the e-rotation through the angle φ about the real axis. Hence, for the image $x + \xi j = \Phi(y + \eta j)$ of a point $y + \eta j, y \in \mathbb{R}, \eta > 0$, of A_S we have

$$\begin{aligned} x &= y + i\eta\sin\varphi = y + i\eta\tanh\delta\,,\\ \xi &= \eta\cos\varphi \qquad = \eta/\cosh\delta\,. \end{aligned}$$

This yields

$$\begin{aligned} \frac{dx\,d\bar{x} + d\xi^2}{\xi^2} &= \frac{dy^2 + \tanh^2\delta\,d\eta^2 + d\eta^2/\cosh^2\delta}{\eta^2/\cosh^2\delta}\\ &= \frac{dy^2 + d\eta^2}{\eta^2}\cosh^2\delta\,. \end{aligned}$$

Hence, the metric on S is that of an hyperbolic plane with curvature $-1/\cosh^2\delta$. The role of the lines is taken over by the *equidistant curves* in which S is intersected by the planes orthogonal to A_S. Their *axes* are the intersection lines of these planes and A_S.

Clearly, any two equidistant surfaces with the same distance δ are congruent in the sense that there are elements of $\mathscr{H}_3^+$ which map one onto the other. Further it is obvious that the subgroup of $\mathscr{H}_3$, whose elements map an equidistant surface S onto itself is identical with the subgroup whose elements map the axial plane A_S and each of the half-spaces bounded by it onto itself and is therefore trivially isomorphic to the isometry group of a plane.

Finally we consider an *equidistant cylinder* C, the locus of the points with a given distance $\varrho > 0$, the *radius* of C, from a line A_C, the axis of C. The generators of C are the equidistant curves with distance ϱ and the common axis A_C. To determine the metric induced on C we may assume A_C to be the line $[0, \infty]$. Then C is an e-cone of revolution. If φ denotes the angle between the axis and a generator, we have again $\tan\varphi = \sinh\varrho$. For a point $x + \xi j$ of C this implies

$$x\bar{x} = \xi^2\sinh^2\varrho\,.$$

As parameters on C we introduce

$$z = x/\xi\,,\quad \zeta = \cosh\varrho\log\xi\,.$$

Then we have $z\bar{z} = \sinh^2\varrho$, so $|z|\arg z$ is the arc length of a parallel circle of C,

and ζ is the arc length of a generator. Since $d(z\bar{z}) = zd\bar{z} + \bar{z}dz = 0$ and $dx = \xi dz + zd\xi$, we obtain

$$dxd\bar{x} = \xi^2 dzd\bar{z} + z\bar{z}d\xi^2 = \xi^2 dzd\bar{z} + \sinh^2 \varrho \, d\xi^2 .$$

Together with $\xi d\zeta = \cosh \varrho \, d\xi$ this yields

$$\frac{dxd\bar{x} + d\xi^2}{\xi^2} = dzd\bar{z} + d\zeta^2 .$$

Hence, the metric induced on C is Euclidean, and C is isometric to an e-cylinder of revolution with e-radius $\sinh \varrho$.

Clearly, given any two cylinders with the same radius, there are elements of $\mathscr{H}_3^+$ mapping one onto the other. The elements of $\mathscr{H}_3$ which map a cylinder C onto itself are precisely those which map its axis A_C onto itself. The subgroup consisting of these elements is obviously isomorphic with the analogous subgroup of the isometry group of Euclidean 3-space.

Notes to Chapter IV

The classification of the isometries of hyperbolic space is of course not new; see for instance Coxeter [7].

Concerning the equidistant surfaces, dealt with in Section 5, it should be observed that there is a deviation from the usual terminology. In the projective model it is convenient to consider the two parts of the locus of points of a given distance from a plane as one surface since they are parts of the same quadric. Here, where orientations are taken into account, it is necessary to consider the parts as different surfaces. Therefore the term "equidistant surface" is used for one of them.

V. Lines

V.1 Line matrices

A proper or improper line is determined by its ends u and u'. If it is proper, that is $u \neq u'$, an orientation of it is fixed by ordering the pair u, u'. Though this determination is extremely simple, the introduction of certain homogeneous coordinates as described below turns out to be much more suitable for the study of the elementary geometry of lines in hyperbolic space. The starting point is the observation that any matrix $\mathbf{l}$ with $\det \mathbf{l} = 1$ and $\operatorname{tr} \mathbf{l} = 0$ determines a half-turn about a proper line, and that this line determines the matrix up to the sign.

A matrix $\mathbf{l} \neq \mathbf{0}$ satisfying $\operatorname{tr} \mathbf{l} = 0$ will be called a *line matrix*. Every class of mutual proportional line matrices determines a line, and conversely. Indeed, if the matrices of a class have non-vanishing determinants, it contains two matrices with determinant 1 and these determine the same half-turn and, thus, a line. If

$$\mathbf{l} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & -l_{11} \end{pmatrix},$$

the ends u, u' of the line are the roots of the equation

$$l_{21} x^2 - 2 l_{11} x - l_{12} = 0 ,$$

and these roots determine $\mathbf{l}$ up to a non-zero factor. If the matrices of a class have vanishing determinants, each of them determines the map which sends all points of the hyperbolic space U (and all points of $\mathbb{C}_\infty$ with one exception) to the same point $u \in \mathbb{C}_\infty$, namely

$$u = l_{11}/l_{21} = -l_{12}/l_{11} .$$

(At least one of these expressions make sense). Since u is the double root of the equation above, the class determines by definition the improper line $[u, u]$. Clearly, every improper line determines a class of matrices of rank 1 and trace 0.

For later use we mention that a matrix $\mathbf{l} \neq \mathbf{0}$ is a line matrix if and only if any one of the obviously equivalent conditions

$$\operatorname{tr} \mathbf{l} = 0 , \quad \mathbf{l}^{\sim} = -\mathbf{l} , \quad \mathbf{l}^2 = -\mathbf{l} \det \mathbf{l}$$

is satisfied (cf. I.3).

Let $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$ be a matrix determining a motion f different from the identity. *Then* $\mathbf{f} - \mathbf{f}^{\sim}$ *is a line matrix determining the (proper or improper) axis of* f. This

follows immediately from the fact that the fixed points in $\mathbb{C}_\infty$ of the transformations determined by $\mathbf{f}$ and by $\mathbf{f}-\mathbf{f}^{\sim}$ are the roots of the same quadratic equation.

Let $\mathbf{f} \in \mathrm{Sl}(2, \mathbb{C})$ again be a matrix determining a motion f different from the identity. *A line L determined by a line matrix* $\mathbf{l}$ *is a normal to the axis of f if and only if*

$$\mathrm{tr}(\mathbf{f}\,\mathbf{l}) = 0 .$$

If the axis of f is proper, it may be assumed to be $[0, \infty]$. Then

$$\mathbf{f} = \begin{pmatrix} f_{11} & 0 \\ 0 & 1/f_{11} \end{pmatrix}, \quad \mathbf{l} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & -l_{11} \end{pmatrix}$$

and thus

$$\mathrm{tr}(\mathbf{f}\,\mathbf{l}) = l_{11}(f_{11} - 1/f_{11}) = 0$$

is equivalent to $l_{11} = 0$, since $f \neq id$ and therefore $f_{11} \neq \pm 1$. Consequently the ends of L are the roots of

$$l_{21}x^2 - l_{12} = 0$$

and hence harmonic with $0, \infty$, also if they coincide with 0 or with ∞. If the axis of f is improper, it may be assumed that

$$\mathbf{f} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

and the condition is then

$$\mathrm{tr}(\mathbf{f}\,\mathbf{l}) = l_{21} = 0 ,$$

hence, that at least one of the ends of L is ∞, which proves the statement in this case.

As a corollary we have:

Let $\mathbf{f}$ *and* $\mathbf{g}$ *be matrices determining motions f and g with distinct axes. Then* $\mathbf{gf} - \mathbf{fg}$ *is a line matrix determining the common normal of the axes of f and g.*

This is a consequence of

$$\mathrm{tr}(\mathbf{f}(\mathbf{gf} - \mathbf{fg})) = 0 , \quad \mathrm{tr}(\mathbf{g}(\mathbf{gf} - \mathbf{fg}) = 0 .$$

Further we have:

Lines L and L′ determined by line matrices $\mathbf{l}$ *and* $\mathbf{l}'$ *are normal to each other if and only if* $\mathrm{tr}(\mathbf{l}'\mathbf{l}) = 0$. *In particular, they are improper and coincide if and only if* $\mathbf{l}'\mathbf{l} = \mathbf{0}$.

The case where at least one of the lines is proper is covered by the statement above. If both are improper, we may assume that $L' = [0,0]$ and $L = [u,u]$, $u \in \mathbb{C}$. Then

$$\mathbf{l}'\mathbf{l} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & -u^2 \\ 1 & -u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u & -u^2 \end{pmatrix},$$

and $\operatorname{tr}(\mathbf{l}'\mathbf{l}) = 0$ is equivalent to $\mathbf{l}'\mathbf{l} = \mathbf{0}$.

Three lines L, L', L'' have a common normal N if and only if line matrices $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ determining them are linearly dependent over $\mathbb{C}$, equivalently, if and only if

$$\operatorname{tr}(\mathbf{l}''\mathbf{l}'\mathbf{l}) = 0\,.$$

A line matrix $\mathbf{n}$ determines a common normal of L, L', L'' if and only if

$$\operatorname{tr}(\mathbf{l}\mathbf{n}) = 0\,, \quad \operatorname{tr}(\mathbf{l}'\mathbf{n}) = 0\,, \quad \operatorname{tr}(\mathbf{l}''\mathbf{n}) = 0\,.$$

These homogeneous linear equations for the elements of $\mathbf{n}$ have a non-zero solution if and only if they, and thus the matrices $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ are linearly dependent. The second statement follows from the fact that the determinant of the system, apart from a non-zero constant factor, equals $\operatorname{tr}(\mathbf{l}''\mathbf{l}'\mathbf{l})$. Indeed $\operatorname{tr}(\mathbf{l}''\mathbf{l}'\mathbf{l})$ is linear in the elements of each of the three matrices and alternating since

$$\operatorname{tr}(\mathbf{l}''\mathbf{l}'\mathbf{l}) = \operatorname{tr}(\mathbf{l}''\mathbf{l}'\mathbf{l})^{\sim} = \operatorname{tr}(\mathbf{l}^{\sim}\mathbf{l}'^{\sim}\mathbf{l}''^{\sim}) = -\operatorname{tr}(\mathbf{l}\mathbf{l}'\mathbf{l}'')$$

and the trace is preserved under cyclic permutations of the matrices.

As a consequence of the statement above we have:

Three proper lines have a common normal if and only if the product of the half-turns about these lines is a half-turn.

Clearly, any four line matrices are linearly dependent.

V.2 Oriented lines

A class of mutual proportional line matrices with non-vanishing determinants contains two matrices with determinant 1. These will be called *normalized line matrices*. It is basic for the treatment of line trigonometry in the following sections that it is possible to assiciate consistently each of these two matrices with one of the orientations of the proper line they determine. Let

$$\mathbf{l} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & -l_{11} \end{pmatrix}, \quad -l_{11}^2 - l_{12}l_{21} = 1\,,$$

be a normalized line matrix. The ends u, u' of the line it determines are the roots of

$$l_{21}x^2 - 2l_{11}x - l_{12} = 0.$$

By definition $\mathbf{l}$ determines the *oriented line* $[u, u']$ provided

$$\begin{array}{lll} u = (l_{11} - i)/l_{21}, & u' = (l_{11} + i)/l_{21} & \text{if} \quad l_{21} \neq 0, \\ u = l_{12}i/2, & u' = \infty & \text{if} \quad l_{21} = 0, l_{11} = i, \\ u = \infty & u' = -l_{12}i/2 & \text{if} \quad l_{21} = 0, l_{11} = -i, \end{array}$$

equivalently, provided

(1) $$\mathbf{l} = \frac{i}{u' - u}\begin{pmatrix} u + u' & -2uu' \\ 2 & -u - u' \end{pmatrix} \quad \text{if} \quad u, u' \in \mathbb{C}$$

and, in accordance with the rules for computation with ∞,

(2) $$\mathbf{l} = i\begin{pmatrix} 1 & -2u \\ 0 & -1 \end{pmatrix} \quad \text{if} \quad u \in \mathbb{C},\ u' = \infty,$$

(3) $$\mathbf{l} = -i\begin{pmatrix} 1 & -2u' \\ 0 & -1 \end{pmatrix} \quad \text{if} \quad u = \infty,\ u' \in \mathbb{C}.$$

This convention is consistent in the sense that it is preserved under motions. More precisely, if $\mathbf{l}$ determines $[u, u']$ and f is a motion determined by $\mathbf{f} \in \mathrm{SL}(2, \mathbb{C})$, then $\mathbf{f}\mathbf{l}\mathbf{f}^{-1}$ determines the line $[f(u), f(u')]$. To see this it is sufficient to consider the motions $x \mapsto -x^{-1}$ and $x \mapsto x + q$, $q \in \mathbb{C}$, determined by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}.$$

Now

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & -l_{11} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -l_{11} & -l_{21} \\ -l_{12} & l_{11} \end{pmatrix}.$$

Since $-l_{12} = 2uu'i/(u' - u) = 2i/(-u'^{-1} + u^{-1})$ if $u, u' \in \mathbb{C} \setminus \{0\}$, the statement is true in this case. If, for instance, $u = 0, u' \in \mathbb{C}$, we have $-l_{12} = 0$, $-l_{21} = 2i/u'$, and $-l_{11} = -i$ in accordance with that the image line is $[\infty, -1/u']$. If, for instance, $u' = \infty$, we have $-l_{12} = 2ui = 2i/(-u^{-1})$, $-l_{21} = 0$, $-l_{11} = -i$ in accordance with that the image line is $[-1/u, 0]$. The remaining cases may be checked by interchanging the roles of u and u'.

Since

$$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & -l_{11} \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} l_{11} + ql_{21} & l_{12} - 2ql_{11} - q^2 l_{21} \\ l_{21} & -l_{11} - ql_{21} \end{pmatrix}$$

the statement is obvious here if $u, u' \in \mathbb{C}$ because

$$l_{21} = 2i/(u' - u) = 2i/(u' + q - u - q)\,.$$

If, for instance, $u' = \infty$, we have $l_{21} = 0$, $l_{11} + q l_{21} = i$, and $l_{12} - 2q\,l_{11} - q^2\,l_{21} = -2ui - 2qi$, in accordance with that the image line is $[u + q, \infty]$. Similarly the case $u = \infty$ may be checked.

To investigate the behaviour of a normalized line matrix under transformation by a reversal it is sufficient to consider the plane-reflection determined by the matrix $\mathbf{1}j$. Let $\mathbf{l}$ be a normalized line matrix determining the oriented line $[u, u']$. The transformed matrix is $\bar{\mathbf{l}}$ (cf. IV.3 (3)). The expressions of $\mathbf{l}$ in terms of u, u' given above show that $\bar{\mathbf{l}}$ determines the oriented line $[\bar{u}', \bar{u}]$ and not, as desired, the image $[\bar{u}, \bar{u}']$ of $[u, u']$. To maintain the convention it is therefore necessary to let correspond to $\mathbf{l}$ not the transformed matrix $\bar{\mathbf{l}}$ but its negative. In general:

If p^* is a reversal determined by the matrix $\mathbf{p}j$ and $\mathbf{l}$ is a normalized line matrix determining the line $[u, u']$, then the image $[p^*(u), p^*(u')]$ of the latter is determined by $-\mathbf{p}\bar{\mathbf{l}}\mathbf{p}^{-1}$.

As an application of the matrix representation of the oriented lines we mention:

Three proper oriented lines L_1, L_2, L_3 determined by the line matrices $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ taken in this order form an orthogonal right-handed frame if and only if

$$\mathbf{l}_3\mathbf{l}_2\mathbf{l}_1 = \mathbf{1}\,. \tag{4}$$

To see the necessity of the condition observe that the statement is invariant under motions and that $\mathscr{H}_3^+$ acts transitively on the set of orthogonal right-handed frames (cf. III.2). It may therefore be assumed that $L_1 = [-1, 1]$, $L_2 = [-i, i]$, $L_3 = [0, \infty]$. The corresponding matrices are

$$\mathbf{l}_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{l}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{l}_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and indeed

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The sufficiency may be seen as follows: Since $\mathbf{l}_n^{\sim} = \mathbf{l}_n^{-1} = -\mathbf{l}_n$, $n = 1, 2, 3$, the condition implies

$$\mathbf{l}_3\mathbf{l}_2 = -\mathbf{l}_1, \quad \mathbf{l}_2\mathbf{l}_1 = -\mathbf{l}_3, \quad \mathbf{l}_1\mathbf{l}_3 = -\mathbf{l}_2,$$

hence

$$\operatorname{tr}(\mathbf{l}_3\mathbf{l}_2) = \operatorname{tr}(\mathbf{l}_2\mathbf{l}_1) = \operatorname{tr}(\mathbf{l}_1\mathbf{l}_3) = 0\,.$$

Consequently any two of L_1, L_2, L_3 intersect orthogonally. They must form a right-handed frame, for otherwise the frame with one of them oppositely oriented would be right-handed, and thus $-\mathbf{l}_3\mathbf{l}_2\mathbf{l}_1 = \mathbf{1}$ because of the necessity of the condition.

Concerning the terminology to be used in the sequel: Two lines will be said to *coincide* if they are identical as point sets, but their orientations need not agree.

If L denotes an oriented line, the same line with the opposite orientation will be denoted by $-L$. If L is improper, we define $-L = L$.

As a first application of the notions introduced we prove the following theorem:

The common normals of opposite sides of a hexagon with all vertices improper have a common normal.

Let $u_1, u_2, \ldots, u_6 \in \mathbb{C}_\infty$ be distinct improper points. We consider the hexagon with side-lines $[u_n, u_{n+1}]$, $n \bmod 6$. It may be assumed that all $u_n \neq \infty$. Then (non-normalized) line matrices for the lines $[u_n, u_{n+1}]$ are

$$\mathbf{s}_{n,n+1} = \begin{pmatrix} u_n + u_{n+1} & -2u_n u_{n+1} \\ 2 & -u_n - u_{n+1} \end{pmatrix}.$$

A line matrix for the common normal of $[u_1, u_2]$ and $[u_4, u_5]$ is

$$\begin{aligned}
\mathbf{a}_{12,45} &= \mathbf{s}_{45}\mathbf{s}_{12} - \mathbf{s}_{12}\mathbf{s}_{45} \\
&= \begin{pmatrix} u_4 + u_5 & -2u_4u_5 \\ 2 & -u_4 - u_5 \end{pmatrix} \begin{pmatrix} u_1 + u_2 & -2u_1u_2 \\ 2 & -u_1 - u_2 \end{pmatrix} \\
&\quad - \begin{pmatrix} u_1 + u_2 & -2u_1u_2 \\ 2 & -u_1 - u_2 \end{pmatrix} \begin{pmatrix} u_4 + u_5 & -2u_4u_5 \\ 2 & -u_4 - u_6 \end{pmatrix} \\
&= 4\begin{pmatrix} u_1u_2 - u_4u_5 & u_4u_5(u_1 + u_2) - u_1u_2(u_4 + u_5) \\ u_1 + u_2 - u_4 - u_5 & u_4u_5 - u_1u_2 \end{pmatrix}.
\end{aligned}$$

For the other two common normals one has, analogously,

$$\mathbf{a}_{23,56} = 4\begin{pmatrix} u_2u_3 - u_5u_6 & u_5u_6(u_2 + u_3) - u_2u_3(u_5 + u_6) \\ u_2 + u_3 - u_5 - u_6 & u_5u_6 - u_2u_3 \end{pmatrix},$$

$$\mathbf{a}_{34,61} = 4\begin{pmatrix} u_3u_4 - u_6u_1 & u_6u_1(u_3 + u_4) - u_3u_4(u_6 + u_1) \\ u_3 + u_4 - u_6 - u_1 & u_6u_1 - u_3u_4 \end{pmatrix}.$$

It has to be shown that these three matrices are linearly dependent. A straightforward calculation shows that indeed

$$(u_6 - u_3)\mathbf{a}_{12,45} + (u_1 - u_4)\mathbf{a}_{23,56} + (u_2 - u_5)\mathbf{a}_{34,61} = \mathbf{0}.$$

V.3 Double crosses

We consider now configurations consisting of an ordered pair of lines L, M, oriented if proper, and a common normal N of them, likewise oriented if proper. The only restriction we require is that no three of the ends of L and M coincide, in other words, that an improper among these lines does not coincide with an end of the other, in particular, that not both are improper and coincide. It is however not excluded that L and M coincide if they are proper. Such a configuration $(L, M; N)$ will be called a *double cross.*

Let

$$L = [u, u'], \quad M = [v, v'], \quad N = [w, w'].$$

Then

$$\mathbb{R}(u, u', w', w) = -1, \quad \mathbb{R}(v', v, w', w) = -1.$$

(Here the convention that the cross ratio is -1 if three but not all of the points coincide has to be used in case improper lines occur, cf. I.4.) Using this one obtains

$$\begin{aligned} \mathbb{R}(u, v, w', w) &= \mathbb{R}(u, v', w', w)\,\mathbb{R}(v', v, w', w) &&= -\,\mathbb{R}(u, v', w', w) \\ &= \mathbb{R}(u, u', w', w)\,\mathbb{R}(u', v, w', w) &&= -\,\mathbb{R}(u', v, w', w) \\ &= -\,\mathbb{R}(u', v', w', w)\,\mathbb{R}(v', v, w', w) &&= \mathbb{R}(u', v', w', w). \end{aligned}$$

With the double cross $(L, M; N)$ we associate an element

$$\mu = \mu(L, M; N)$$

of the extended complex cylinder $\mathbb{A}_\infty$ (cf. I.2), called the *width* of $(L, M; N)$, defined by

$$\begin{aligned} \exp \mu &= \mathbb{R}(u, v, w', w) = -\,\mathbb{R}(u, v', w', w) = -\,\mathbb{R}(u', v, w', w) \\ &= \mathbb{R}(u', v', w', w). \end{aligned} \tag{1}$$

If L and M are parallel or coincide, the four points in one of these cross ratios coincide. This has then to be neglected. If all three lines are proper, μ is the displacement of the motion f with fixed points w, w', taken in this order, for which $f(u) = v$ (cf. IV.2). Hence $\operatorname{Re}\mu$ is the distance from L to M provided with a sign in accordance with the orientation of N, and $\operatorname{Im}\mu$ is the angle from L to M in the obvious sense. This interpretation of μ is also valid if there are improper lines among L, M, N. Indeed, if L and M are parallel, thus N improper, $\mu = 0$ provided L and M are both oriented towards N or both away from N and $\mu = \pi i$ in the other cases. If N is proper and L or M or both are improper, we have $\mu = +\infty$ or $\mu = -\infty$ according as the direction from L towards M agrees or does not agree with the orientation of N.

Let $\mathbf{l}, \mathbf{m}, \mathbf{n}$ be matrices determining L, M, N, normalized in accordance with the orientations of the proper ones among these lines. Then

$$
\begin{aligned}
&\cosh \mu = -\frac{1}{2}\operatorname{tr}(\mathbf{ml}) &&\text{whenever} \quad L, M \quad \text{are proper},\\
(2)\qquad &\sinh \mu = \frac{1}{2} i \operatorname{tr}(\mathbf{nml}) &&\text{whenever} \quad L, M \quad \text{are proper},\\
&\tanh \mu = -i\frac{\operatorname{tr}(\mathbf{nml})}{\operatorname{tr}(\mathbf{ml})} &&\text{whenever} \quad N \quad \text{is proper}.
\end{aligned}
$$

(Comparison with $\cosh \mu = \frac{1}{2}\operatorname{tr}\mathbf{f}^2$ where $\mathbf{f} \in \mathrm{SL}(2,\mathbb{C})$ determines the motion f mentioned above, which was proved in IV.2, shows that the matrix **ml** which determines f^2 equals $-\mathbf{f}^2$.) To prove the statements in the case where all three of the lines are proper, we may assume that $N = [0, \infty]$, $L = [-1, 1]$. Then $M = [-v, v]$ where $v = \exp \mu$ since $\not{R}(1, v, \infty, 0) = v$. The corresponding matrices are

$$\mathbf{l} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 0 & vi \\ i/v & 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and thus

$$\operatorname{tr}(\mathbf{ml}) = -(v + 1/v), \quad \operatorname{tr}(\mathbf{nml}) = -i(v - 1/v)$$

from which the relations follow in the case considered. If N is improper we may assume that it is $[0, 0]$, that L is $[0, \infty]$ or $[\infty, 0]$, and that M is $[0, 1]$ or $[1, 0]$. Then

$$\mathbf{l} = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{m} = \pm\begin{pmatrix} i & 0 \\ i & -i \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence, in accordance with $\mu = 0$ or πi, we have $\operatorname{tr}(\mathbf{nml}) = 0$, and $\operatorname{tr}(\mathbf{ml}) = -2$ if the signs above agree, otherwise $\operatorname{tr}(\mathbf{ml}) = 2$. That the expression for $\tanh \mu$ remains valid if L or M is improper is also easily checked. It may be assumed that $N = [0, \infty]$ or $[\infty, 0]$ and for instance $L = [0, 0]$, $M = [-1, 1]$ or $[\infty, \infty]$, thus

$$\mathbf{l} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{n} = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

This yields

$$\operatorname{tr}(\mathbf{ml}) = i \quad \text{or} \quad 1, \quad \pm\operatorname{tr}(\mathbf{nml}) = -1 \quad \text{or} \quad i,$$

hence

$$\frac{\operatorname{tr}(\mathbf{nml})}{\operatorname{tr}(\mathbf{ml})} = \mp i$$

in accordance with $\mu = \pm\infty$.

With notations introduced above we have

$$(3)\qquad \not{R}(u, u', v, v') = \tanh^2\frac{\mu}{2} = \frac{\cosh \mu - 1}{\cosh \mu + 1}.$$

To prove this, assume first that the ends u, u', v, v' of L and M are distinct. Then $w \neq w'$ and indeed (cf. I.4)

$$\begin{aligned}\not R(u,u',v,v') &= \not R(u,w,v,v')\,\not R(w,w',v,v')\,\not R(w',u',v,v')\\ &= -\,\not R(u,w,v,w')\,\not R(u,w,w',v')\,\not R(w',u',v,w)\,\not R(w',u',w,v')\\ &= -(1-e^{-\mu})\,\frac{-e^{\mu}}{-e^{\mu}-1}\cdot\frac{1}{1+e^{\mu}}(1-e^{\mu}) = \left(\frac{e^{\mu}-1}{e^{\mu}+1}\right)^2 .\end{aligned}$$

If coincidences of the ends occur, the relation is easily verified directly. If $u = u'$ or/and $v = v'$ we have $\not R(u,u',v,v') = 1$ in accordance with $\mu = \pm\infty$. If $u = v$ or/and $u' = v'$, the cross ratio equals 0 and $\mu = 0$. If $u = v'$ or/and $u' = v$, the cross ratio equals ∞ and $\mu = \pi i$.

Four proper oriented lines L_0, L_1, L_2, L_3 together with common normals of any two of them determine six double crosses. There is a relation between the widths of these. To derive it consider the normalized line matrices $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ determining the lines. Obviously, they are linearly dependent over $\mathbb{C}$, so

$$\sum_{m=0}^{3} p_m \mathbf{l}_m = \mathbf{0}$$

with some $p_m \in \mathbb{C}$, not all zero. Multiplying by $\mathbf{l}_n$ and taking traces, we obtain

$$\sum_{m=0}^{3} p_m \operatorname{tr}(\mathbf{l}_m \mathbf{l}_n) = 0\,, \quad n = 0,1,2,3\,.$$

Consequently,

$$\det(\operatorname{tr}(\mathbf{l}_m \mathbf{l}_n))_{m,n=0,1,2,3} = 0\,.$$

Let μ_{mn} denote the width of the double cross $(L_m, L_n; N_{mn})$, where N_{mn} denotes a common normal of L_m and L_n, oriented arbitrarily if proper. Then

$$\operatorname{tr}(\mathbf{l}_m \mathbf{l}_n) = \begin{cases} -2 & \text{if } \; m = n\,,\\ -2\cosh\mu_{mn} & \text{if } \; n \neq n\,,\end{cases}$$

and the relation in question is

$$\begin{vmatrix} 1 & \cosh\mu_{01} & \cosh\mu_{02} & \cosh\mu_{03}\\ \cosh\mu_{01} & 1 & \cosh\mu_{12} & \cosh\mu_{13}\\ \cosh\mu_{02} & \cosh\mu_{12} & 1 & \cosh\mu_{23}\\ \cosh\mu_{03} & \cosh\mu_{13} & \cosh\mu_{23} & 1 \end{vmatrix} = 0\,.$$

V.4 Transversals

Let $(L, M; N)$ be a double cross and $\mathbf{l}, \mathbf{m}, \mathbf{n}$ matrices corresponding to the lines L, M, N, normalized for the proper ones. We assume now that L and M do not coincide, so $\mathbf{l}$ and $\mathbf{m}$ are linearly independent. A line T is called a *transversal* of the double cross if it intersects N orthogonally. The line matrices representing the transversals are precisely the linear combinations of $\mathbf{l}$ and $\mathbf{m}$. For each (non-oriented) transversal T, with the exception of M, there is precisely one line matrix of the form

$$\mathbf{t} = \mathbf{l} - r\mathbf{m}\,, \quad r \in \mathbb{C}\,,$$

which determines it. If L and M are proper, hence $\mathbf{l}$ and $\mathbf{m}$ normalized, r admits of a geometrical interpretation. It is then called the *ratio in which $T = T_r$ divides the double cross.*

We consider first the case where N is proper and $\mathbf{n}$ normalized. Using $\mathbf{l}^2 = -\mathbf{1}$, $\mathbf{m}^2 = -\mathbf{1}$, $\operatorname{tr}(\mathbf{nlm}) = \operatorname{tr}(\mathbf{nlm})^{\sim} = -\operatorname{tr}(\mathbf{mln})$, we obtain

$$\frac{\operatorname{tr}(\mathbf{ntl})}{\operatorname{tr}(\mathbf{ntm})} = \frac{\operatorname{tr}(\mathbf{n}(\mathbf{l} - r\mathbf{m})\mathbf{l})}{\operatorname{tr}(\mathbf{n}(\mathbf{l} - r\mathbf{m})\mathbf{m})} = \frac{r\operatorname{tr}(\mathbf{nml})}{\operatorname{tr}(\mathbf{nlm})} = r\,.$$

If T_r is proper, $\mathbf{t}$ may be normalized without changing the left-hand side. With an arbitrarily chosen orientation of T_r we then have

$$r = \frac{\sinh \mu(L, T_r; N)}{\sinh \mu(M, T_r; N)}\,.$$

The right-hand side may be written

$$\frac{\sinh\left[\mu(L, M; N) + \mu(M, T_r; N)\right]}{\sinh \mu(M, T_r; N)} = \sinh \mu(L, M; N) \coth \mu(M, T_r; N) + \cosh \mu(L, M; N)\,,$$

and the last expression makes also sense if $\mu(M, T_r; N)$ is $-\infty$ or $+\infty$, that is, if T_r is improper and thus coincides with the initial or terminal point of N. The expression then takes the values

$$r = \exp\left[-\mu(L, M; N)\right]\,, \quad r = \exp \mu(L, M; N)$$

and these are indeed the correct ones, namely the roots of

$$\begin{aligned}\det(\mathbf{l} - r\mathbf{m}) &= 1 - r\operatorname{tr}(\mathbf{lm}^{\sim}) + r^2 \\ &= 1 - 2r\cosh \mu(L, M; N) + r^2 = 0\,.\end{aligned}$$

We consider now a double cross with N improper, L and M proper. It is necessary to distinguish between two cases:
1) L and M are both oriented towards N or both away from N.
2) One of them is oriented towards N, the other away from N.

Since r changes sign if the orientation of one of L and M is reversed, we may assume (in case 1) that L und M are oriented towards N. Let $N = [w', w']$, $L = [u, w']$, $M = [v, w']$ and $T_r = [w, w']$. Then we claim that

$$r = \not R(u, v, w, w') .$$

This holds if T_r is improper, that is, $w = w'$, since $r = 1$ is the double root of

$$\det(\mathbf{l} - r\mathbf{m}) = 1 - 2r + r^2 = 0 .$$

Let now T_r be proper, $r \neq 1$. Since the statement is invariant under motions, we may assume $w' = \infty$. Then we have to consider

$$\not R(u, v, w, \infty) = \frac{w-u}{w-v} .$$

The matrices determining L, M, and T_r are

$$\mathbf{l} = \begin{pmatrix} -i & 2ui \\ 0 & i \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} -i & 2vi \\ 0 & i \end{pmatrix}$$

$$\mathbf{t} = \mathbf{l} - r\mathbf{m} = \begin{pmatrix} -i(1-r) & 2(u-rv)i \\ 0 & i(1-r) \end{pmatrix} .$$

This shows that

$$w = \frac{u - rv}{1 - r}$$

and consequently

$$\frac{w-u}{w-v} = r$$

as claimed. The latter relation may be interpreted invariantly. Let $N = [w', w']$ with $w' \in \mathbb{C}_\infty$ arbitrarily, and let S be a horosphere with centre w'. The metric induced on S is Euclidean (cf. IV.5), and S may therefore be considered as a complex plane with ∞ at w'. If now the complex coordinates of the points of intersection of L, M, T_r with S are denoted by u, v, w, respectively, the relation above holds, since it does for $w' = \infty$ and the complex structure of S is preserved under motions.

In the case 2) the relations above hold with r replaced by $-r$ if u, v, w in the cross ratio denote the ends different from w' of the lines L, M, T_r, respectively.

Let $(L, M; N)$ again be a double cross with L and M proper and distinct as point sets. We consider the transversals dividing it in the ratios $r = -1$ and $r = 1$.

If N is proper, $r = -1$ is equivalent to

$$\sinh \mu(L, T_{-1}; N) = -\sinh \mu(M, T_{-1}; N),$$

hence

$$\mu(L, T_{-1}; N) = -\mu(M, T_{-1}; N) \quad \text{or}$$
$$\mu(L, T_{-1}; N) = \mu(M, T_{-1}; N) - \pi i.$$

The second equality has however to be excluded since it would imply that L and M coincide (with opposite orientations). The first one shows that the half-turn about T_{-1} maps L onto M with the right orientation. For $r = 1$ we have

$$\sinh \mu(L, T_1; N) = \sinh \mu(M, T_1; N),$$

hence

$$\mu(L, T_1; N) = \mu(M, T_1; N) \quad \text{or} \quad \mu(L, T_1; N) = -\mu(M, T_1; N) + \pi i.$$

Here the first equality has to be excluded since it implies $L = M$. The second one shows that the half-turn about T_1 maps L onto $-M$, that is, M with the orientation reversed.

If N is improper and L and M are concordantly oriented, T_1 is improper and coincides with N. With any one of the definitions of u, v, w above one has $w = \frac{1}{2}(u + v)$ for $r = -1$, which shows that the half-turn about T_{-1} also in this case maps L onto M with the right orientation. If L and M are oppositely oriented, T_{-1} also in this case maps L onto M with the right orientation. If L and M are oppositely oriented, T_{-1} is improper and coincides with N, and the half-turn about T_1 maps L onto $-M$.

The transversals T_{-1} and T_1 will be called the *concordant bisector* and the *reverse bisector* of the double cross $(L, M; N)$, respectively. They intersect orthogonally at the midpoint of the segment on N joining L and M and bisect the angles between L and M in the obvious sense. Line matrices determining them are $\mathbf{l} + \mathbf{m}$ and $\mathbf{l} - \mathbf{m}$, respectively.

V.5 Pencils and bundles of lines

An *elliptic pencil* consists of the lines in a plane which pass through a proper point of it. A *parabolic pencil* consists of the lines in a plane with a common end. A *hyperbolic pencil* consists of the lines in a plane which intersect a line in this plane orthogonally. A pencil is determined by any two of its lines.

Let $(L, M; N)$ be a double cross with L and M distinct and proper. For $\mu = \mu(L, M; N)$ we then have (cf. V.3(4))

$$\cosh \mu = \not< - \frac{1}{2} \operatorname{tr}(\mathbf{ml}),$$

where $\mathbf{l}$ and $\mathbf{m}$ denote the normalized line matrices determining L and M. As mentioned in V.3, these lines are coplanar if and only if

$$(1) \qquad \operatorname{tr}(\mathbf{ml}) \in \mathbb{R}.$$

They intersect, are parallel, or are ultraparallel according as

$$(2) \qquad |\operatorname{tr}(\mathbf{ml})| \begin{cases} < 2 \\ = 2 \\ > 2 \end{cases}.$$

The lines of the pencil containing two coplanar lines L und M are transversals of $(L, M; N)$. Hence they are determined by normalized line matrices of the form

$$\mathbf{t}_r = \frac{\mathbf{l} - r\mathbf{m}}{(\det(\mathbf{l} - r\mathbf{m}))^{1/2}}, \quad r \in \mathbb{C}.$$

A line T_r determined by such a matrix belongs to the pencil if and only if

$$\operatorname{tr}(\mathbf{t}_r \mathbf{l}) = \frac{-2 - r\operatorname{tr}(\mathbf{ml})}{(1 + r\operatorname{tr}(\mathbf{ml}) + r^2)^{1/2}} \in \mathbb{R},$$

$$\operatorname{tr}(\mathbf{t}_r \mathbf{m}) = \frac{\operatorname{tr}(\mathbf{ml}) - 2r}{(1 + r\operatorname{tr}(\mathbf{ml}) + r^2)^{1/2}} \in \mathbb{R},$$

Considering the quotient of these expressions one infers that the conditions are equivalent to

$$(3) \qquad r \in \mathbb{R}, \quad 1 + r\operatorname{tr}(\mathbf{ml}) + r^2 > 0.$$

From the preceding we conclude in particular:

Normalized line matrices $\mathbf{k}, \mathbf{l}, \mathbf{m}$ *determine lines belonging to the same pencil if and only if they are linearly dependent over* $\mathbb{R}$ *and*

$$\operatorname{tr}(\mathbf{ml}), \quad \operatorname{tr}(\mathbf{lk}), \quad \operatorname{tr}(\mathbf{km}) \in \mathbb{R}.$$

If L and M intersect, the inequality (3) is satisfied for all $r \in \mathbb{R}$ since $|\operatorname{tr}(\mathbf{ml}| < 2$. Hence, the matrices $\mathbf{m}$ and $\mathbf{l} - r\mathbf{m}$, $r \in \mathbb{R}$, determine the lines of the pencil. The interpretation (1) of r takes here the form

$$r = \frac{\sin \varphi(L, T_r)}{\sin \varphi(M, T_r)},$$

where $\varphi(L, T_r)$ and $\varphi(M, T_r)$ denote the angles from L and M to T_r provided with signs in accordance with an arbitrary orientation of N.

If L and M are parallel and, say, concordantly oriented, we have $\operatorname{tr}(\mathbf{ml}) = -2$ and (3) is satisfied for all $r \in \mathbb{R} \setminus \{1\}$. Hence, the matrices $\mathbf{m}$ and $\mathbf{l} - r\mathbf{m}$, $r \neq 1$, determine the lines of the pencil. For $r = 1$ the improper line coinciding with the common end of L and M is obtained. It may be considered as belonging to the pencil. The interpretation (2) of r may here be described as follows. Choose an oriented horocycle with centre at this improper point and lying in the plane of the pencil. Let u, v, and w_r denote the points at which it is intersected by L, M, and T_r, respectively. Then

$$r = \frac{d(\mathrm{u}, \mathrm{w}_r)}{d(\mathrm{v}, \mathrm{w}_r)}, \tag{5}$$

where $d(\mathrm{u}, \mathrm{w}_r)$ and $d(\mathrm{v}, \mathrm{w}_r)$ denote the lengths, provided with signs, of the horocycle arcs from u and v to w_r.

If L and M are ultraparallel and, say, concordantly oriented, and N is oriented from L towards M, we obtain from V.3(2)

$$\operatorname{tr}(\mathbf{ml}) = -2\cosh\delta,$$

where $\delta = \delta(L, M) > 0$ denotes the distance from L to M. Hence, condition (3) requires here

$$1 - 2r\cosh\delta + r^2 = (r - \exp(-\delta))(r - \exp\delta) > 0,$$
$$r < \exp(-\delta) \quad \text{or} \quad r > \exp\delta.$$

The matrices $\mathbf{l} - r\mathbf{m}$, for these values of r, and $\mathbf{m}$ determine the lines of the pencil. The interpretation (1) of r may here be written

$$r = \frac{\sinh\delta(L, T_r)}{\sinh\delta(M, T_r)}. \tag{6}$$

For $r = \exp(-\delta)$ and $r = \exp\delta$ the improper lines coinciding with the ends of N are obtained. They may be considered as belonging to the pencil.

For every r' satisfying

$$\exp(-\delta) < r' < \exp\delta \tag{7}$$

$i\mathbf{t}_{r'}$ is a normalized line matrix. The trace of the product of any two of these matrices is real and any three of them are linearly dependent over $\mathbb{R}$. Since $r' = \exp(-\delta)$ and $r' = \exp\delta$ yield the improper lines coinciding with the ends of N, the matrices $i\mathbf{t}_{r'}$ for the values of r' satisfying (7) determine the lines of a hyperbolic pencil with common normal N. Since the reverse bisector of L and M is obtained for $r' = 1$, which satisfies (7), the plane of this pencil is orthogonal to that of the pencil containing L and M.

We define four types of bundles of proper lines.

A *planar line bundle* consists of the lines of a plane. An *elliptic line bundle* consists of the lines through a proper point. A *parabolic line bundle* consists of the lines with a common end. A *hyperbolic line bundle* consists of the lines orthogonal to a plane.

A common property of line bundles is that every two lines of a bundle are coplanar, but its lines do not belong to one pencil. It will turn out that, conversely, a set of lines every two of which are coplanar is contained in a line bundle.

Let K, L, M be distinct, proper, pairwise coplanar lines which do not belong to a pencil. If all three of the lines are coplanar, they clearly belong to a planar line bundle. Suppose that they are not coplanar. Then M is the intersection of a plane through K and a plane through L. Consequently, if K and L intersect, their common point must also lie on M. Hence, K, L, M belong to an elliptic bundle. If K and L are parallel, the same argument shows that M is parallel to them with the same end. Hence, K, L, M belong to a parabolic bundle. If K and L are ultraparallel, there is a unique plane intersecting them orthogonally, namely the plane through their common normal and orthogonal to their plane. The planes through K or L, and therefore also the intersection M of two of them, will be orthogonal to that plane. This shows that K, L, M belong to a hyperbolic bundle.

Let $\mathbf{k}, \mathbf{l}, \mathbf{m}$ be normalized line matrices determining K, L, M, respectively. The matrices are linearly independent over $\mathbb{R}$, for otherwise the lines would belong to a pencil. Further, the matrices satisfy (cf. (1))

$$(8) \qquad \operatorname{tr}(\mathbf{lk}), \quad \operatorname{tr}(\mathbf{km}), \quad \operatorname{tr}(\mathbf{ml}) \in \mathbb{R},$$

since the lines are pairwise coplanar. According to the statements above, the lines belong to a unique bundle if and only if the matrices satisfy these conditions.

We are going to discuss how the type of the bundle containing K, L, M can be determined by means of the matrices $\mathbf{k}, \mathbf{l}, \mathbf{m}$. Let k, l, m denote the half-turns about K, L, M respectively, and let δ be the displacement of the motion $m \circ l \circ k$. Then we have (cf. IV.2(1))

$$\operatorname{tr}^2(\mathbf{mlk}) = 2 + 2\cosh\delta .$$

If K, L, M are coplanar, $m \circ l \circ k$ is a skrew motion with $\operatorname{Re}\delta \neq 0$ and $\operatorname{Im}\delta = \pi$. Hence, $\cosh\delta < -1$, so

$$(9) \qquad \operatorname{tr}^2(\mathbf{mlk}) < 0$$

if K, L, M belong to a planar bundle. In the three other cases we have

$$(10) \qquad \operatorname{tr}^2(\mathbf{mlk}) \geq 0 .$$

Indeed, $m \circ l \circ k$ is a rotation, a limit rotation, or a translation, so δ is purely imaginary or real and thus $\cosh\delta \geqq -1$. In the parabolic case, and only in it,

$n \circ m \circ l$ is a half-turn, in accordance with that K, L, M have a common improper normal. Hence, $\delta = \pi i$ or, equivalently, $\operatorname{tr}^2(\mathbf{mlk}) = 0$ is characteristic for this case.

For a further discussion the relation

$$(11) \qquad 2\operatorname{tr}^2(\mathbf{nml}) = \begin{vmatrix} 2 & -\operatorname{tr}(\mathbf{ml}) & -\operatorname{tr}(\mathbf{lk}) \\ -\operatorname{tr}(\mathbf{ml}) & 2 & -\operatorname{tr}(\mathbf{km}) \\ -\operatorname{tr}(\mathbf{lk}) & -\operatorname{tr}(\mathbf{km}) & 2 \end{vmatrix} = \Delta$$

is useful. It is obtained by applying I.3(6) to the matrices $\mathbf{a} = -\mathbf{ml}$, $\mathbf{b} = -\mathbf{km}$, $\mathbf{c} = -\mathbf{lk}$. Indeed, $\operatorname{tr}(\mathbf{abc}) = -\operatorname{tr}(\mathbf{mlk})^2 = -\operatorname{tr}^2(\mathbf{mlk}) + 2$ by I.3(3).

From (9) and (11) we infer that K, L, M belong to a planar bundle if and only if $\operatorname{tr}(\mathbf{lk})$, $\operatorname{tr}(\mathbf{km})$, $\operatorname{tr}(\mathbf{ml})$ are real numbers for which the determinant Δ in (11) is negative.

Application of the first equation in V.3(2) to the double crosses consisting of the pairs (K, L), (M, K), (L, M) and their common normals shows that the traces, in addition to being real, satisfy the following conditions in the other cases. K, L, M belong to an elliptic bundle if and only if

$$|\operatorname{tr}(\mathbf{lk})| < 2\,, \quad |\operatorname{tr}(\mathbf{km})| < 2\,, \quad |\operatorname{tr}(\mathbf{ml})| < 2$$

and $\Delta > 0$. The three lines belong to a parabolic bundle if and only if

$$|\operatorname{tr}(\mathbf{lk})| = |\operatorname{tr}(\mathbf{km})| = |\operatorname{tr}(\mathbf{ml})| = 2$$

and $\Delta = 0$. (The latter condition is easily seen to be equivalent to $\operatorname{tr}(\mathbf{lk})\operatorname{tr}(\mathbf{km})\operatorname{tr}(\mathbf{ml}) = -8$.) The three lines belong to a hyperbolic pencil if and only if

$$|\operatorname{tr}(\mathbf{lk})| > 2\,, \quad |\operatorname{tr}(\mathbf{km})| > 2\,, \quad |\operatorname{tr}(\mathbf{ml})| > 2$$

and $\Delta > 0$. (The latter condition implies that the product of the traces is negative.)

Let K, L, M again be pairwise coplanar lines which do not belong to a pencil, and let $\mathbf{k}, \mathbf{l}, \mathbf{m}$ be normalized line matrices determining them. We are going to show:

The normalized line matrices determining the lines of the bundle containing K, L, M are precisely the linear combinations

$$(12) \qquad p\mathbf{k} + q\mathbf{l} + r\mathbf{m}$$

with real coefficients p, q, r satisfying

$$(13) \qquad \begin{aligned} \det(p\mathbf{k} + q\mathbf{l} + r\mathbf{m}) &= p^2 + q^2 + r^2 - pq\operatorname{tr}(\mathbf{lk}) - pr\operatorname{tr}(\mathbf{km}) - qr\operatorname{tr}(\mathbf{ml}) \\ &= 1. \end{aligned}$$

In the proof the parabolic case has to be treated separately. In the other cases $\mathbf{k}, \mathbf{l}, \mathbf{m}$ are linearly independent over $\mathbb{C}$ and any line N is determined by a normalized matrix $\mathbf{n}$ in the form (12) with complex coefficients p, q, r satisfying (13). Now N belongs to the bundle if and only if it is coplanar with each of K, L, M, thus if and only if

$$\begin{aligned}
\operatorname{tr}(\mathbf{nk}) &= -2p \quad + q \operatorname{tr}(\mathbf{lk}) + r \operatorname{tr}(\mathbf{mk}) \in \mathbb{R}\,,\\
\operatorname{tr}(\mathbf{nl}) &= p \operatorname{tr}(\mathbf{kl}) - 2q \quad + r \operatorname{tr}(\mathbf{ml}) \in \mathbb{R}\,,\\
\operatorname{tr}(\mathbf{nm}) &= p \operatorname{tr}(\mathbf{km}) + q \operatorname{tr}(\mathbf{lm}) - 2r \in \mathbb{R}\,.
\end{aligned}$$

These conditions are obviously satisfyed if p, q, r are real, and only then because for a given $\mathbf{n}$ they form the unique solution of a system of linear equations with real coefficients and non-vanishing determinant $-\Delta$. If K, L, M belong to a parabolic bundle, $\mathbf{k}, \mathbf{l}, \mathbf{m}$ are linearly dependent over $\mathbb{C}$ since the three lines have a common improper normal. A normalized line matrix $\mathbf{n}$ determining a line of the bundle is a complex linear combination of two of the given ones, $\mathbf{n} = s\mathbf{k} + t\mathbf{l}$, say. If, for the sake of simplicity, we assume all lines of the bundle to be oriented concordantly, the trace of the product of any two normalized matrices determining lines of it equals -2. Multiplying the expression for $\mathbf{n}$ by $\mathbf{n}$ and taking traces, one obtains therefore $s + t = 1$. It is easily checked that $\det \mathbf{n} = 1$ for any $s, t \in \mathbb{C}$ with sum 1. In particular, we have $\mathbf{m} = a\mathbf{k} + (1 - a)\mathbf{l}$, where a is not real since K, L, M do not belong to a pencil. For any real r we may therefore write

$$\mathbf{n} = (s - ra)\mathbf{k} + (1 - s - (1 - a)r)\,\mathbf{l} + r\mathbf{m}\,.$$

If we choose $r = \operatorname{Im} s / \operatorname{Im} a$, the three coefficients are real. (In case K, L, M are not concordantly oriented, certain changes of signs have to be made.) This proves the statement in the present case.

Finally we observe that the quadratic form $\det(p\mathbf{k} + q\mathbf{l} - r\mathbf{m})$ in (10) is positive definite in the elliptic case, so that to any triplet of real numbers, not all zero, there is a proportional one satisfying (13). In the parabolic case the form is positive semidefinite, in fact, it equals $(p + q + r)^2$ if it is assumed as above that the lines are concordantly oriented. If it vanishes, (12) determines the improper line represented by the common end of the lines of the bundle. In the other cases the quadratic form is indefinite. Consider pairs each consisting of a hyperbolic bundle and a planar bundle such that the lines of the former are the normals of the plane of the latter. Then we have: If K, L, M determine a bundle of such a pair, the matrices

$$(14) \qquad (r\mathbf{k} + s\mathbf{l} + t\mathbf{m})i\,, \quad r, s, t \in \mathbb{R}\,, \quad \det(r\mathbf{k} + s\mathbf{l} + t\mathbf{m}) = -1\,,$$

determine the lines of the other one. This is a consequence of the fact that the trace of the product of a matrix (12) satisfying (13) and a matrix (14) is purely imaginary. Indeed, the first equation in V.3 (2) shows that the width of the double cross consisting of the two lines and their common normal has imaginary part $\pm\pi/2$.

Notes to Chapter V

Several types of coordinates for the line in hyperbolic space have been used. If one works with the projective model, it is natural to use Plücker's line coordinates, cf. Coolidge [3], 110–115. Another possibility was noted by Klein [11]: Consider the complex projective plane with a given polarity. There exists a bijective correspondence between the lines of hyperbolic space and the pairs each consisting of a point and its polar in the complex plane. This is also mentioned in the book of Coolidge [3], 118, and the paper of Coxeter [7]. Klein and Coxeter use it to derive "Pascal's Theorem" for hexagons with improper vertices (proved here in Section 2). A thorough treatment of such an approach, dealing also with oriented lines, is to be found in Study [28]. The obvious way to determine a line in hyperbolic space, namely by the pair of its ends, is discussed in the notes to Chapter VI.

The determination of a line in the half-space model by a (2×2)-matrix of trace 0, which is used here, seems to be new. Its advantages are that it is easy to take care of the orientations and that metrical relations between several lines can be expressed simply in terms of traces.

VI. Right-angled Hexagons

VI.1 Right-angled hexagons and pentagons

The configuration to be studied is a cyclically ordered 6-tuplet of lines S_n, $n \bmod 6$, oriented if proper, such that for every $n \bmod 6$ the lines S_n and S_{n+1} are normal to each other, but not the same improper line, and the lines S_{n-1} and S_{n+1} do not coincide. If these conditions are satisfied $(S_1, S_2, S_3, S_4, S_5, S_6) = (S_n, n \bmod 6)$ is called a *right-angled hexagon* and S_n, $n \bmod 6$, its *side-lines*.

Neither the side-lines S_1, S_3, S_5 nor the side-lines S_2, S_4, S_6 of a right-angled hexagon have a common normal; for if, for instance, S_1, S_3, S_5 had one, it would be unique and S_2, S_4, S_6 would coincide with it in contradiction to the condition above. Further, if a side-line S_n is improper, the adjacent ones, S_{n-1}, S_{n+1} are proper. Hence, at most three of the side-lines are improper.

Let S_1, S_3, S_5 be lines which have no common normal and such that an improper among them does not coincide with an end of one of the others. If S_2, S_4, S_6 are the common normals of S_1 and S_3, of S_3 and S_5, of S_5 and S_1, respectively, and the proper ones are oriented arbitrarily, then $(S_n, n \bmod 6)$ is a right-angled hexagon. Indeed not two of S_2, S_4, S_6 can coincide since S_1, S_3, S_5 have no common normal and, for instance, S_1 and S_2 cannot be the same improper line, for otherwise S_1 would coincide with an end of S_3.

Let $\mathbf{s}_n$, $n \bmod 6$, be line matrices determining the side-lines of a right-angled hexagon. For the proper ones of these lines they are supposed to be normalized in accordance with the orientations. They satisfy the following conditions for $n \bmod 6$:

$$\operatorname{tr}(\mathbf{s}_{n+1}\mathbf{s}_n) = 0\,, \quad \mathbf{s}_{n+1}\mathbf{s}_n \neq \mathbf{0}\,,$$
$$\mathbf{s}_{n-1} \quad \text{and} \quad \mathbf{s}_{n+1} \quad \text{are linearly independent}\,.$$

Conversely, six line-matrices $\mathbf{s}_n$ satisfying these conditions determine the side-lines of a right-angled hexagon.

Three line matrices $\mathbf{s}_1, \mathbf{s}_3, \mathbf{s}_5$, *normalized if regular, determine side-lines* S_1, S_3, S_5 *of a right-angled hexagon if and only if they are linearly independent and in case* $\det \mathbf{s}_n = 0$ *for an* $n = 1, 3, 5$, *they satisfy* $\operatorname{tr}(\mathbf{s}_n \mathbf{s}_{n-2}) \neq 0$, $\operatorname{tr}(\mathbf{s}_{n+2}\mathbf{s}_n) \neq 0$.

This is obvious since the first condition is equivalent to the non-existence of a common normal to S_1, S_3, S_5, and the second says that an improper among these

lines is not normal to one of the other two, that is, it does not coincide with an end of one of the others.

Line matrices for the three other side-lines S_2, S_4, S_6, which are determined up to orientation, are $\mathbf{s}_3\mathbf{s}_1 - \mathbf{s}_1\mathbf{s}_3$, $\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5$, $\mathbf{s}_1\mathbf{s}_5 - \mathbf{s}_5\mathbf{s}_1$, respectively. These matrices are indeed different from $\mathbf{0}$. For the cases where the two line matrices involved are regular, see the remark at the end of IV.2. If one or both of them are singular, one easily checks, assuming one to be $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, that they commute only if they are proportional.

Three successive side-lines of a right-angled hexagon form a double-cross $(S_{n-1}, S_{n+1}; S_n)$. Its width $\sigma_n \in \mathbb{A}$ will briefly be called the n-th *side* of the hexagon. If $\mathbf{s}_n$, $n \bmod 6$, are line matrices, normalized if proper, determining the side-lines, we have (cf. V.3(2)), observing that $\mathbf{s}_n\mathbf{s}_{n+1} = -\mathbf{s}_{n+1}\mathbf{s}_n$:

$$\cosh\sigma_n = -\frac{1}{2}\operatorname{tr}(\mathbf{s}_{n+1}\mathbf{s}_{n-1}) \qquad \text{if} \quad S_{n-1}, S_{n+1} \quad \text{are proper,}$$

$$\sinh\sigma_n = -\frac{i}{2}\operatorname{tr}(\mathbf{s}_{n+1}\mathbf{s}_n\mathbf{s}_{n-1}) \neq 0 \quad \text{if} \quad S_{n-1}, S_n, S_{n+1} \quad \text{are proper,}$$

$$\coth\sigma_n = -i\frac{\operatorname{tr}(\mathbf{s}_{n+1}\mathbf{s}_{n-1})}{\operatorname{tr}(\mathbf{s}_{n+1}\mathbf{s}_n\mathbf{s}_{n-1})} \qquad \text{if} \quad S_n \quad \text{is proper}.$$

If five cyclically ordered lines S_n, $n \bmod 5$, satisfy the same conditions as the side-lines of a right-angled hexagon, $(S_1, S_2, S_3, S_5) = (S_n, n \bmod 5)$ is called a *right-angled pentagon* with side-lines S_n. At most two of these may be improper. The *sides* σ_n, $n \bmod 5$, are defined as for an hexagon. If $\mathbf{s}_n$, $n \bmod 5$, are line-matrices determining the S_n, normalized in accordance with the orientations for the proper ones, then the relations above are valid with $n \bmod 5$.

We list the special types of right-angled hexagons and pentagons which are relevant for the study of the geometry in a hyperbolic plane P. Clearly, the side-lines have to be contained in or normal to P.

If all side-lines of a hexagon are proper and lie in P, it is also a *right-angled hexagon* in P. Considering only the segments of the side-lines which join consecutive vertices, we have in P then either a convex right-angled hexagon or one with self-intersection. To see this, observe that in this case any two of S_1, S_3, S_5 and any two of S_2, S_4, S_6 must be ultraparallel. If none of S_1, S_3, S_5 separates the other two, the same holds of S_2, S_4, S_6; for, if S_4, say, would separate S_2 and S_6, it would have to intersect S_1 which is impossible since the points of intersection of S_4 with S_3 and S_5 lie in the same half-plane bounded by S_1. In this case the hexagon with the intersections of neighbouring lines S_n as vertices is convex because these vertices lie on the same side or on each of the lines S_n. If one of S_1, S_3, S_5 separates the two others, it follows from what has been said that the same holds for S_2, S_4, S_6. In this case the hexagon is obviously self-intersecting.

If precisely one of the side-lines, S_1 say, is normal to P, possibly improper on the horizon of P, the lines S_3 and S_5 are concurrent or ultraparallel. If the intersection of S_1 and P, for the sake of brevity called the point S_1, lies between S_3 and S_5, we obtain in P a simple *pentagon with four right angles*. It is convex since the angle at S_1 is less than π. If one of S_3, S_5 separates the other from S_1, a self-intersecting pentagon with four right angles results. In the transitional cases where the point S_1 lies on S_3 or S_5 the pentagon degenerates: one of the vertices of a right angle coincides with S_1.

If two non-opposite side-lines, S_1 and S_3 say, are normal to P, possibly improper, while the others are proper and lie in P, we have in P a *quadrangle with two adjacent right angles*, convex if the points S_1 and S_3 lie on the same side of S_5, self-intersecting if S_5 separates S_1 and S_3. In the transitional cases where S_1 or S_3 or both lie on S_5 the quadrangle degenerates.

If two opposite side-lines, S_1 and S_4 say, are normal to P, possibly improper, there results a *quadrangle with two opposite right angles*. Here S_3 and S_5 intersect at the point S_4 or are parallel with common end S_4. If they intersect, let S_3' and S_5' denote the lines in P through S_4 and orthogonal to S_3 and S_5, respectively. The lines S_3, S_5, S_3', S_5' bound eight angular regions. (These coincide pairwise if $S_3 \perp S_5$). If the point S_1 lies in one of those bounded by S_3 and S_5 or by S_3' and S_5', the quadrangle is convex. If S_1 lies in one of the others, it is self-intersecting. It degenerates if S_1 lies on one of the lines S_3, S_5, S_3', S_5'. If S_3 and S_5 are parallel, the quadrangle is convex, self-intersecting, or degenerate according as S_1 lies in the interior, in the exterior, or on the boundary of the strip bounded by S_3 and S_5.

If three of the side-lines, S_1, S_3, S_5 say, are normal to P, possibly improper, a *triangle* results.

Special hexagons of interest are also those whose side-lines have a proper point $\mathrm{c} \in U$ in common. Let C denote the sphere with centre c and radius Arsinh 1 $= \log(1 + \sqrt{2})$. As shown in III.5, C is isometric with the Euclidean unit sphere. The points at which the positive half-lines S_1, S_3, S_5 with initial point c intersect C are the vertices of a *spherical triangle*, and those at which the positive half-lines S_2, S_4, S_6 intersect C are the vertices of a polar triangle. The sides σ_n are i times the spherical lengths of the sides of these triangles provided with signs depending on the orientations.

VI.2 Trigonometric relations for right-angled hexagons

Relations between the sides σ_n of a right-angled hexagon or pentagon will be obtained as trace relations between products of the matrices s_n determining the side-lines S_n. We summarize the properties of these matrices to be used.

For $n \bmod 6$ or $n \bmod 5$

$$\begin{aligned} &\operatorname{tr}\mathbf{s}_n = 0\,, &&\operatorname{tr}(\mathbf{s}_{n+1}\mathbf{s}_n) = 0\,,\\ &\mathbf{s}_n^{\sim} = -\mathbf{s}_n\,, &&\mathbf{s}_{n+1}\mathbf{s}_n = -\mathbf{s}_n\mathbf{s}_{n+1}\,,\\ &\mathbf{s}_n^2 = -\mathbf{1}\det\mathbf{s}_n\,, &&(\mathbf{s}_{n+1}\mathbf{s}_n)^2 = -\mathbf{1}\det\mathbf{s}_{n+1}\det\mathbf{s}_n\,,\\ &\operatorname{tr}(\mathbf{s}_{n+1}\mathbf{s}_n\mathbf{s}_{n-1}) \neq 0\,, \end{aligned}$$

since then S_{n-1}, S_n, S_{n+1} have no common normal. The matrices corresponding to proper side-lines will tacitly be assumed to be normalized in accordance with the orientations. Hence the determinants occurring above have the value 1 or 0.

Further the trace relation (I.3(2))

$$\operatorname{tr}(\mathbf{ab}) + \operatorname{tr}(\mathbf{a}^{\sim}\mathbf{b}) = \operatorname{tr}\mathbf{a}\operatorname{tr}\mathbf{b}$$

will be used frequently without reference.

Suppose that side-lines S_2 and S_3 of a pentagon or hexagon are proper. Then

$$\begin{aligned} \operatorname{tr}(\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2)\operatorname{tr}(\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1) &= \operatorname{tr}(\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1) - \operatorname{tr}(\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1)\\ &= -2\operatorname{tr}(\mathbf{s}_4\mathbf{s}_1)\,. \end{aligned} \tag{1}$$

In the case of a right-angled pentagon with S_1, S_2, S_3, S_4 proper we have $\operatorname{tr}(\mathbf{s}_4\mathbf{s}_1) = -2\cosh\sigma_5$, and the relation with the subscripts 1, 2, 3, 4, 5 changed to $n+1$, $n+2$, $n-2$, $n-1$, n, where n is taken mod 5, yields (cf. V.4):

The sides σ_n of a right-angled pentagon (S_n, $n \bmod 5$) with at most one improper side-line satisfy

$$\cosh\sigma_n = -\sinh\sigma_{n-2}\sinh\sigma_{n+2} \tag{2}$$

for every $n \bmod 5$ if all side-lines are proper, otherwise for the number n of the improper side-line.

To interprete $\operatorname{tr}(\mathbf{s}_4\mathbf{s}_1)$ in the case of a right-angled hexagon we consider the double cross $(S_1, S_4; A_{14})$ where $A_{14} = A_{41}$, an *altitude line* of the hexagon, is a common normal of S_1 and S_4, oriented arbitrarily if proper. (If S_1 and S_4 coincide, and hence are proper, A_{14} may be any one of their proper normals.) The width $\alpha_{14} = \alpha_{41}$ of the double cross $(S_1, S_4; A_{14})$ will be called an *altitude* of the hexagon. Analogously the other altitude lines and altitudes $A_{25} = A_{52}$, $\alpha_{25} = \alpha_{52}$ and $A_{36} = A_{63}$, $\alpha_{36} = \alpha_{63}$ are defined.

The relation (1) between traces may now be written

$$\cosh\alpha_{14} = -\sinh\sigma_2\sinh\sigma_3\,.$$

If all side-lines of the hexagon are proper, we also have

$$\cosh\alpha_{14} = \cosh\alpha_{41} = -\sinh\sigma_5\sinh\sigma_6\,.$$

Since any cyclical permutation of the subscripts $1, 2, \ldots, 6$ is permitted, we obtain the *"Law of Sines"*.

The sides σ_n *of a right-angled hexagon* $(S_n, n \bmod 6)$ *with all side-lines proper satisfy*

$$\text{(3)} \qquad \frac{\sinh \sigma_1}{\sinh \sigma_4} = \frac{\sinh \sigma_3}{\sinh \sigma_6} = \frac{\sinh \sigma_5}{\sinh \sigma_2}.$$

Let now S_1, S_2, S_3, S_4, S_5 be side-lines of a right-angled pentagon or hexagon, and suppose that S_2 and S_4 are proper. Then we have

$$\text{(4)} \qquad \begin{aligned} &\operatorname{tr}(\mathrm{s}_5\mathrm{s}_4\mathrm{s}_3)\operatorname{tr}(\mathrm{s}_4\mathrm{s}_2)\operatorname{tr}(\mathrm{s}_3\mathrm{s}_2\mathrm{s}_1) \\ &\qquad = \operatorname{tr}(\mathrm{s}_5\mathrm{s}_4\mathrm{s}_3)\left[\operatorname{tr}(\mathrm{s}_4\mathrm{s}_2\mathrm{s}_3\mathrm{s}_2\mathrm{s}_1) + \operatorname{tr}(\mathrm{s}_2\mathrm{s}_4\mathrm{s}_3\mathrm{s}_2\mathrm{s}_1)\right] \\ &\qquad = 2\operatorname{tr}(\mathrm{s}_5\mathrm{s}_4\mathrm{s}_3)\operatorname{tr}(\mathrm{s}_4\mathrm{s}_3\mathrm{s}_1) \\ &\qquad = 2\operatorname{tr}(\mathrm{s}_5\mathrm{s}_4\mathrm{s}_3\mathrm{s}_4\mathrm{s}_3\mathrm{s}_1) - 2\operatorname{tr}(\mathrm{s}_3\mathrm{s}_4\mathrm{s}_5\mathrm{s}_4\mathrm{s}_3\mathrm{s}_1) \\ &\qquad = -2\operatorname{tr}(\mathrm{s}_5\mathrm{s}_1)\det\mathrm{s}_3 - 2\operatorname{tr}(\mathrm{s}_3\mathrm{s}_5\mathrm{s}_3\mathrm{s}_1) \\ &\qquad = -4\operatorname{tr}(\mathrm{s}_5\mathrm{s}_1)\det\mathrm{s}_3 - 2\operatorname{tr}(\mathrm{s}_3\mathrm{s}_5)\operatorname{tr}(\mathrm{s}_3\mathrm{s}_1). \end{aligned}$$

If $S_1, S_2, \ldots, S_5$ are side-lines of a pentagon, $\operatorname{tr}(\mathrm{s}_5\mathrm{s}_1) = 0$, and division of the relation by $\operatorname{tr}(\mathrm{s}_5\mathrm{s}_4\mathrm{s}_3)$ and $\operatorname{tr}(\mathrm{s}_3\mathrm{s}_2\mathrm{s}_1)$ yields

$$\cosh \sigma_3 = -\coth \sigma_2 \coth \sigma_4.$$

Hence we may conclude:

The sides σ_n *of a right-angled pentagon* $(S_n, n \bmod 5)$ *satisfy*

$$\text{(5)} \qquad \cosh \sigma_n = -\coth \sigma_{n-1} \coth \sigma_{n+1}$$

for all $n \bmod 5$ *such that* S_{n-1} *and* S_{n+1} *are proper.*

Consider now a hexagon, and assume that S_1, S_3, S_5 also are proper. Then the relation above may be written

$$\cosh \sigma_6 = \cosh \sigma_2 \cosh \sigma_4 + \sinh \sigma_2 \sinh \sigma_4 \cosh \sigma_3.$$

Hence, we have the *"Law of Cosines"*:

The sides σ_n *of a right-angled hexagon* $(S_n, n \bmod 6)$ *with at most one improper side-line satisfy*

$$\text{(6)} \qquad \cosh \sigma_n = \cosh \sigma_{n-2} \cosh \sigma_{n+2} + \sinh \sigma_{n-2} \sinh \sigma_{n+2} \cosh \sigma_{n+3}$$

for all $n \bmod 6$ *if all side-lines are proper, otherwise for the number* n *of the improper side-line.*

The Law of Cosines may be given other useful forms. For any choice of the halfs of the sides we have

$$\begin{aligned}\cosh^2\frac{\sigma_{n+3}}{2} &= \frac{1}{2}(\cosh\sigma_{n+2}+1) = \frac{\cosh\sigma_n - \cosh(\sigma_{n-2}-\sigma_{n+2})}{2\sinh\sigma_{n-2}\sinh\sigma_{n+2}} \\ &= \frac{\sinh\frac{1}{2}(-\sigma_{n-2}+\sigma_n+\sigma_{n+2})\sinh\frac{1}{2}(\sigma_{n-2}+\sigma_n-\sigma_{n+2})}{\sinh\sigma_{n-2}\sinh\sigma_{n+2}},\end{aligned} \tag{7}$$

similarly

$$\sinh^2\frac{\sigma_{n+3}}{2} = -\frac{\sin\frac{1}{2}(\sigma_{n-2}+\sigma_n+\sigma_{n+2})\sinh\frac{1}{2}(\sigma_{n-2}-\sigma_n+\sigma_{n+2})}{\sinh\sigma_{n-2}\sinh\sigma_{n+2}} \tag{8}$$

and consequently

$$\tanh^2\frac{\sigma_{n+3}}{2} = -\frac{\sinh\frac{1}{2}(\sigma_{n-2}+\sigma_n+\sigma_{n+2})\sinh\frac{1}{2}(\sigma_{n-2}-\sigma_n+\sigma_{n+2})}{\sinh\frac{1}{2}(-\sigma_{n-2}+\sigma_n+\sigma_{n+2})\sinh\frac{1}{2}(\sigma_{n-2}+\sigma_n-\sigma_{n+2})}. \tag{9}$$

Further we prove the *"Law of Cotangents"*:

The sides σ_n of a right-angled hexagon $(S_n, n \bmod 6)$ *with at most one improper side-line satisfy*

$$\coth\sigma_n\sinh\sigma_{n+2} + \cosh\sigma_{n+1}\cosh\sigma_{n+2} + \sinh\sigma_{n+1}\coth\sigma_{n+3} = 0 \tag{10}$$

for all $n \bmod 6$ *if all side-lines are proper, otherwise for the values of n for which S_{n-1} or S_{n+4} is improper.*

If all side-lines are proper, this may be obtained by suitable eliminations from the Law of Cosines and the Law of Sines. We give a direct proof covering also the other cases.

In terms of traces the statement with $n = 1$ may be written

$$\frac{2\,\mathrm{tr}(\mathbf{s}_2\mathbf{s}_6)}{\mathrm{tr}(\mathbf{s}_2\mathbf{s}_1\mathbf{s}_6)}\mathrm{tr}(\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2) + \frac{2\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_3)}{\mathrm{tr}(\mathbf{s}_5\mathbf{s}_4\mathbf{s}_3)}\mathrm{tr}(\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1) = \mathrm{tr}(\mathbf{s}_3\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_4\mathbf{s}_2)$$

whenever S_1, S_2, S_3, S_4 are proper, while S_5 or S_6 may be improper. From (1) with permissable changes of the subscripts we obtain

$$\begin{aligned}\mathrm{tr}(\mathbf{s}_5\mathbf{s}_4\mathbf{s}_3)\,\mathrm{tr}(\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2) &= -2\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_2).\\ \mathrm{tr}(\mathbf{s}_3\mathbf{s}_2\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_2\mathbf{s}_1\mathbf{s}_6) &= -2\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_6).\end{aligned}$$

Further we have

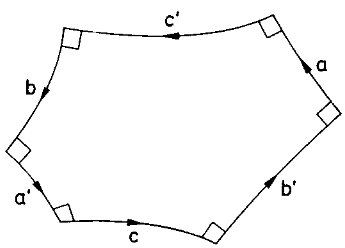
$$\begin{aligned}
&\mathrm{tr}(s_3 s_1)\,\mathrm{tr}(s_2 s_1 s_6)\,\mathrm{tr}(s_4 s_2)\,\mathrm{tr}(s_5 s_4 s_3)\\
&\quad = [\mathrm{tr}(s_3 s_1 s_2 s_1 s_6) + \mathrm{tr}(s_1 s_3 s_2 s_1 s_6)] \cdot [\mathrm{tr}(s_4 s_2 s_5 s_4 s_3) + \mathrm{tr}(s_2 s_4 s_5 s_4 s_3)]\\
&\quad = 4\,\mathrm{tr}(s_3 s_2 s_6)\,\mathrm{tr}(s_2 s_5 s_3) = -4\,\mathrm{tr}(s_2 s_6 s_2 s_5) - 4\,\mathrm{tr}(s_6 s_3 s_5 s_3)\\
&\quad = -4\,\mathrm{tr}(s_2 s_6)\,\mathrm{tr}(s_2 s_5) - 4\,\mathrm{tr}(s_6 s_3)\,\mathrm{tr}(s_5 s_3)\,.
\end{aligned}$$

The statement is an immediate consequence of these three relations.

VI.3 Trigonometric relations for polygons in a plane

We add a list of those special cases of the Law of Sines and the Law of Cosines which are relevant for the geometry of the hyperbolic plane (cf. VI.1). A plane P is assumed to be oriented (in the figures below counter-clockwise) and the side-lines orthogonal to P to be directed towards the positive side of P (in the figures towards the reader). The side-lines in P are supposed to be oriented as indicated in the figures. The notations are chosen as follows: a, a', b, b', c, c' denote the positive lengths of sides and non-negative numbers $\alpha, \alpha', \beta, \gamma$ less than π interior angles of the polygons in question. Numbers in brackets refer to formulae in the preceding section.

3.1 The convex right-angled hexagon



The sides are

$$\begin{aligned}
\sigma_1 &= a + \pi i\,, & \sigma_3 &= b + \pi i\,, & \sigma_5 &= c + \pi i\,,\\
\sigma_4 &= a' + \pi i\,, & \sigma_6 &= b' + \pi i\,, & \sigma_2 &= c' + \pi i\,.
\end{aligned}$$

Hence, (3) takes the form

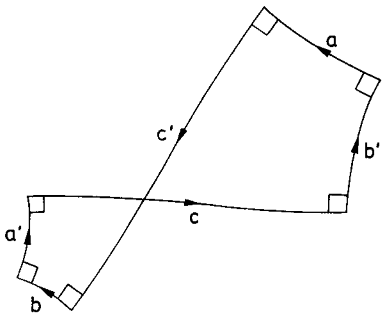
$$\frac{\sinh a}{\sinh a'} = \frac{\sinh b}{\sinh b'} = \frac{\sinh c}{\sinh c'}\,,$$

and (6) yields

$$\cosh a = -\cosh b \cosh c + \sinh b \sinh c \cosh a'$$

and five analogous relations.

3.1′ The self-intersecting right-angled hexagon



For the sides σ_n we have here

$$\sigma_1 = a + \pi i\,, \quad \sigma_3 = b + \pi i\,, \quad \sigma_5 = c\,,$$
$$\sigma_4 = a' + \pi i\,, \quad \sigma_6 = b' + \pi i\,, \quad \sigma_2 = c'\,.$$

Hence, (3) takes the same form as for the convex hexagon, and (6) yields

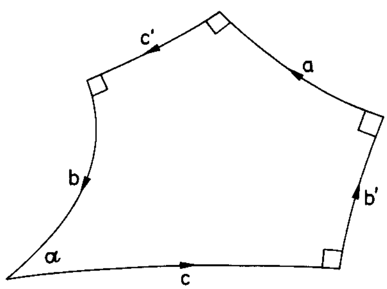
$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cosh \alpha'$$

and the analogues for $\cosh a'$, $\cosh b$, $\cosh b'$,

$$\cosh c = \cosh a \cosh b + \sinh a \sinh b \cosh c'$$

and the analogue for $\cosh c'$.

3.2 The convex pentagon with four right angles



The sides σ_n are here

$$\sigma_1 = a + \pi i, \qquad \sigma_3 = b - \frac{\pi}{2} i, \quad \sigma_5 = c - \frac{\pi}{2} i,$$
$$\sigma_4 = (\pi - \alpha) i, \quad \sigma_6 = b' + \pi i, \quad \sigma_2 = c' + \pi i.$$

Hence, (3) takes the form

$$\frac{\sinh a}{\sin \alpha} = \frac{\cosh b}{\sinh b'} = \frac{\cosh c}{\sinh c'}$$

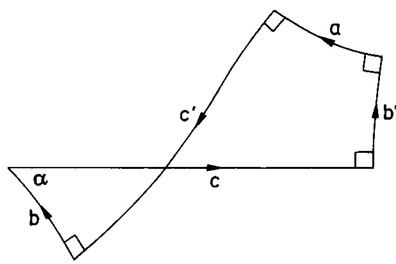
and (6) yields

$$\begin{aligned}
\cosh a &= \sinh b \sinh c - \cosh b \cosh c \cos \alpha, \\
\cos \alpha &= -\cosh b' \cosh c' + \sinh b' \sinh c' \cosh a, \\
\sinh b &= -\cosh a \sinh c + \sinh a \cosh c \cosh b', \\
\cosh b' &= -\cos \alpha \cosh c' + \sin \alpha \sinh c' \sinh b
\end{aligned}$$

and the analogues for $\sinh c$ and $\cosh c'$. The expression for $\cos \alpha$ makes sense and is valid if $\alpha = 0$ and hence $b = c = +\infty$.

Relations between the sides of a right-angled pentagon are obtained for $\alpha = \frac{\pi}{2}$ or by specializing (2) and (4).

3.2′ The self-intersecting pentagon with four right angles



The sides σ_n are here

$$\sigma_1 = a + \pi i, \qquad \sigma_3 = b + \frac{\pi}{2} i, \quad \sigma_5 = c - \frac{\pi}{2} i$$
$$\sigma_4 = (\alpha - \pi) i, \quad \sigma_6 = b' + \pi i, \quad \sigma_2 = c'.$$

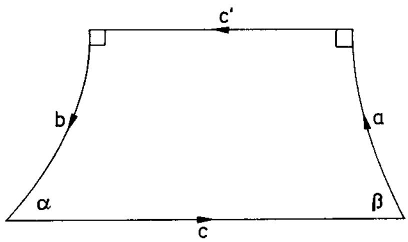
Hence, (3) takes the same form as for the convex pentagon, and (6) yields

$$\begin{aligned}
\cosh a &= -\sinh b \sinh c + \cosh b \cosh c \cos \alpha, \\
\cos \alpha &= \cosh b' \cosh c' - \sinh b' \sinh c' \cosh a, \\
\sinh b &= \cosh a \sinh c - \sinh a \cosh c \cosh b',
\end{aligned}$$

$$\cosh b' = \cos\alpha \cosh c' - \sin\alpha \sinh c' \sinh b\,,$$
$$\sinh c \;= \cosh a \sinh b + \sinh a \cosh b \cosh c'$$
$$\cosh c' = \cos\alpha \cosh b' + \sin\alpha \sin b' \sinh c\,.$$

The expression for $\cos\alpha$ makes again sense and is valid for $\alpha = 0$, $b = c = +\infty$.

3.3 The convex quadrangle with two adjacent right angles



The sides σ_n are here

$$\sigma_1 = a - \frac{\pi}{2}i\,, \quad \sigma_3 = b - \frac{\pi}{2}i\,, \quad \sigma_5 = c\,,$$
$$\sigma_4 = (\pi - \alpha)i\,, \quad \sigma_6 = (\pi - \beta)i\,, \quad \sigma_2 = c' + \pi i\,.$$

Hence, (3) takes the form

$$\frac{\cosh a}{\sin\alpha} = \frac{\cosh b}{\sin\beta} = \frac{\sinh c}{\sinh c'}\,,$$

and (6) yields

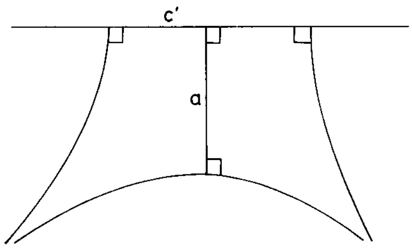
$$\sinh a = \sinh b \cosh c - \cosh b \sinh c \cos\alpha\,,$$
$$\cos\alpha \;= -\cos\beta \cosh c' + \sin\beta \sinh c' \sinh a\,,$$

and the analogues for $\sinh b$ and $\cos\beta$, further

$$\cosh c \;= -\sinh a \sinh b + \cosh a \cosh b \cosh c'\,,$$
$$\cosh c' = -\cos\alpha \cos\beta + \sin\alpha \sin\beta \cosh c\,.$$

The expression for $\cos\alpha$ or for $\cos\beta$ makes sense and is valid for $\alpha = 0$, $b = c = +\infty$ or $\beta = 0$, $a = c = +\infty$, respectively.

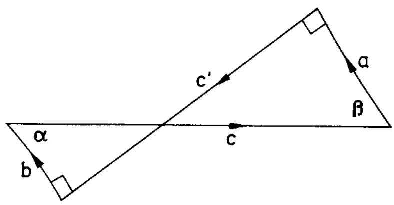
Putting $\beta = \dfrac{\pi}{2}$ or by specializing (2) and (5) one obtains the relations between the non-right angle and the sides of a quadrangle with three right angles. For $\alpha = 0$, $\beta = \dfrac{\pi}{2}$ one obtains



$\sinh a \sinh c' = 1$.

This is the relation between the distance $2c'$ of two ultraparallel lines and the length a of the orthogonal projection of one of them onto the other.

3.3′ The self-intersecting quadrangle with two adjacent right angles



The sides σ_n are here

$$\sigma_1 = a - \frac{\pi}{2} i, \quad \sigma_3 = b + \frac{\pi}{2} i, \quad \sigma_5 = c,$$
$$\sigma_4 = (\alpha - \pi) i, \quad \sigma_6 = (\pi - \beta) i, \quad \sigma_2 = c'.$$

Hence (3) takes the same form as for the convex quadrangle, and (6) yields

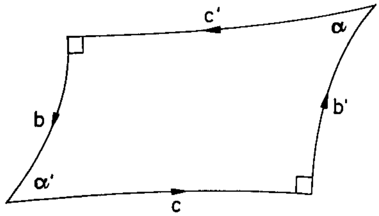
$$\begin{aligned} \sinh a &= -\sinh b \cosh c + \cosh b \sinh c \cos\alpha, \\ \cos\alpha &= \cos\beta \cosh c' - \sin\beta \sinh c' \sinh a, \end{aligned}$$

and the analogues for $\sinh b$ and $\cos\beta$, further

$$\begin{aligned} \cosh c &= \sinh a \sinh b + \cosh a \cosh b \cosh c', \\ \cosh c' &= \cos\alpha \cos\beta + \sin\alpha \sin\beta \cosh c. \end{aligned}$$

Also here $\alpha = 0$ or $\beta = 0$ is admissible in the expressions for $\cos\alpha$ or $\cos\beta$, respectively.

3.4 The convex quadrangle with two opposite right angles



The sides σ_n are here

$$\sigma_1 = (\pi - \alpha)i\,, \quad \sigma_3 = b - \frac{\pi}{2}i\,, \quad \sigma_5 = c - \frac{\pi}{2}i\,,$$

$$\sigma_4 = (\pi - \alpha')i\,, \quad \sigma_6 = b' - \frac{\pi}{2}i\,, \quad \sigma_2 = c' - \frac{\pi}{2}i\,.$$

Hence, (3) takes the form

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{\cosh b}{\cosh b'} = \frac{\cosh c}{\cosh c'}\,,$$

and (6) yields

$$\cos\alpha = \sinh b \sinh c - \cosh b \cosh c \cos\alpha'\,,$$

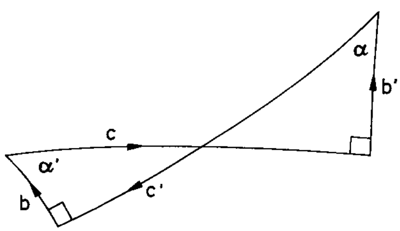
and the analogue for $\cos\alpha'$, further

$$\sinh b = -\cos\alpha \sinh c + \sin\alpha \cosh c \sinh b'$$

and the analogues for $\sinh b'$, $\sinh c$, $\sinh c'$. The expression for $\cos\alpha$ makes sense and is valid if $\alpha = 0$ and hence $b' = c' = +\infty$.

If one of α and α' equals $\frac{\pi}{2}$ one obtains, of course, again the relations for a quadrangle with three right angles.

3.4′ The self-intersecting quadrangle with two opposite right angles



The sides are here

$$\sigma_1 = (\pi - \alpha)i\,, \quad \sigma_3 = b + \frac{\pi}{2}i\,, \quad \sigma_5 = c - \frac{\pi}{2}i\,,$$

$$\sigma_4 = (\alpha' - \pi)i\,, \quad \sigma_6 = b' - \frac{\pi}{2}i\,, \quad \sigma_2 = c' + \frac{\pi}{2}i\,.$$

Hence, (3) takes the same form as for the convex quadrangle, and (6) yields

$$\cos\alpha = -\sinh b \sinh c + \cosh b \cosh c \cos\alpha'\,,$$

and the analogue for $\cos\alpha'$,

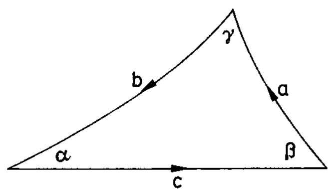
$$\sinh b = \cos\alpha \sinh c - \sin\alpha \cosh c \sinh b'\,,$$

and the analogue for $\sinh b'$,

$$\sinh c = \cos\alpha \sinh b + \sin\alpha \cosh b \sinh c'\,,$$

and the analogue for $\sinh c'$. For $\alpha = 0$ see the remark above.

3.5 The triangle



The sides σ_n are here

$$\sigma_1 = a\,, \qquad \sigma_3 = b\,, \qquad \sigma_5 = c\,,$$
$$\sigma_4 = (\pi - \alpha)i\,, \quad \sigma_6 = (\pi - \beta)i\,, \quad \sigma_2 = (\pi - \gamma)i\,.$$

Hence, (3) takes the form

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$$

and (6) yields

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha\,,$$

and the analogues for $\cosh b$ and $\cosh c$, further

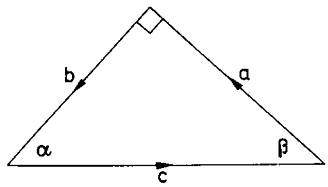
$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a$$

and the analogues for $\cos\beta$ and $\cos\gamma$. The expression for $\cos\alpha$, or $\cos\beta$, or $\cos\gamma$ remains valid if $\alpha = 0$ or $\beta = 0$ or $\gamma = 0$, respectively.

Relations for the right-angled triangle are obtained for $\gamma = \frac{\pi}{2}$, or, more directly, by specializing (2) and (5). These applied to

$$\sigma_1 = c\,,\ \sigma_2 = (\pi - \beta) i,\ \sigma_3 = a - \frac{\pi}{2} i\,,\ \sigma_4 = b - \frac{\pi}{2} i\,,\ \sigma_5 = (\pi - \alpha)\, i$$

yield



$$\begin{aligned}
\cosh c &= \cosh a \cosh b\,, \\
\cos\alpha &= \sin\beta \cos a, \qquad \cos\beta = \sin\alpha \cosh b\,, \\
\sinh a &= \sin\alpha \sinh c, \qquad \sinh b = \sin\beta \sinh c\,,
\end{aligned}$$

$$\begin{aligned}
\cosh c &= \cot\alpha \cot\beta\,, \\
\cos\alpha &= \tanh b \coth c, \quad \cos\beta = \tanh a \coth c, \\
\sinh a &= \cot\beta \tanh b, \quad \sinh b = \cot\alpha \tanh a.
\end{aligned}$$

For $\alpha = 0$, $b = c = +\infty$ we obtain the mutually equivalent relations

$$\sin\beta \cosh a = 1\,, \quad \cos\beta = \tanh a\,, \quad \sinh a = \cot\beta$$

which determine the *parallel angle* $\beta = \Pi$ (a) as a function of the distance a. Given a line L and a point p at distance a from L, then $\Pi(a)$ is the acute angle between the normal to L through p and one of the half-lines from p parallel to L.

From the relations above we draw conclusions concerning the angles of the plane polygons considered.

Consider first a right-angled triangle with the notations above. Since the right-hand sides of the equations for $\cos\alpha$ and $\cos\beta$ are positive, α and β are acute. Now $\cot\alpha \cot\beta = \cosh c > 1$ gives

$$\tan\beta < \cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$$

and hence $\alpha + \beta < \frac{\pi}{2}$.

Consider now a non-right-angled triangle ABC and assume the notations chosen such that $c \geq b \geq a$. Then the altitude from C is in the interior of ABC since the distance of its foot from A and B are smaller than b and a, respectively,

and thus smaller than c. Application of the previous result to the two right-angled triangles into which ABC is divided yields:

The angles α, β, γ of a triangle satisfy

$$\alpha + \beta + \gamma < \pi .$$

Dividing a convex quadrangle with two right angles into two triangles, one infers:

The sum of the non-right angles of a convex quadrangle with two right angles is less than π.

The non-right angle of a quadrangle with three right angles is acute.

Further it is easily seen that simple non-convex right angled hexagons, pentagons with four right angles, and quadrangles with two right angles do not exist, as already used in the enumeration of the plane polygons in V.4.

Finally we mention the specializations of (3) and (6) to spherical trigonometry.

3.6 The spherical triangle

As mentioned at the end of V.4, a right-angled hexagon whose sidelines all pass through a proper point determines a spherical triangle and a polar triangle of it. The lengths a, b, c of the sides and the angles α, β, γ of the triangle have to be considered as arbitrary real numbers mod 2π. The lengths of the sides of the polar triangle are $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$. For the sides of the hexagon we then have

$$\begin{aligned} \sigma_1 &= ai , & \sigma_3 &= bi , & \sigma_5 &= ci , \\ \sigma_4 &= (\pi - \alpha) i , & \sigma_6 &= (\pi - \beta) i , & \sigma_2 &= (\pi - \gamma) i . \end{aligned}$$

Hence, (3) takes the form

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma} ,$$

and (6) yields

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha ,$$

and the analogues for $\cos b$ and $\cos c$,

$$\cos \alpha = - \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a ,$$

and the analogues for $\cos \beta$ and $\cos \gamma$.

VI.4 Determination of a hexagon by three of its sides

We deal now with the question to what extent a right-angled hexagon (S_n, $n \bmod 6$) is determined up to motions if three of its sides σ_n are given. Recall

that the definition of $(S_n,\, n \bmod 6)$ implies that if a side σ_n equals 0 or πi, the side-line S_n is improper and consequently $\sigma_{n-1} = \pm\infty$ and $\sigma_{n+1} = \pm\infty$. If $\sigma_n = \pm\infty$, then at least one of σ_{n-1} and σ_{n+1} equals 0 or πi.

Three cases have to be considered: I. Three adjacent sides are given. II. Three pairwise non-adjacent sides are given. III. Two adjacent sides and the opposite of one of them are given.

I. *Given*

$$\sigma_1 \in \mathbb{A}_\infty \setminus \{0, \pi i\}\,, \quad \sigma_2 \in \mathbb{A} \setminus \{0, \pi i\}\,, \quad \sigma_3 \in \mathbb{A}_\infty \setminus \{0, \pi i\}\,,$$

there exists a right-angled hexagon $(S_n,\, n \bmod 6)$ *with these sides. It is uniquely determined up to the orientation of* S_5 *if* S_5 *is proper.*

Since σ_2 is finite and different from 0 and πi, the side-lines S_1, S_2, S_3 must be proper. Considering that motions are permitted, we may choose S_1 and thereafter S_2 normal to S_1 arbitrarily. Then S_3 is uniquely determined by σ_2 and does not coincide with S_1. Now S_6 and S_4, possibly improper, are determined by σ_1 and σ_3, respectively. None of them coincides with S_2 since $\sigma_1 \neq 0, \pi i$ and $\sigma_3 \neq 0, \pi i$, and this implies that they do not coincide. Hence σ_5 is determined up to orientation as the common normal of S_4 and S_6.

Admissible values of $\sigma_1, \sigma_2, \sigma_3$ which have been excluded by the assumptions are $\sigma_2 = 0$ or πi, implying $\sigma_1 = \pm\infty$ and $\sigma_3 = +\infty$, further $\sigma_2 = \pm\infty$, implying $\sigma_1 = 0$ or πi, or/and $\sigma_3 = 0$ or πi. In these cases there are, as is easily seen, infinitely many hexagons satisfying the requirement.

II. *Given*

$$\sigma_1 \in \mathbb{A}\,, \quad \sigma_3 \in \mathbb{A}\,, \quad \sigma_5 \in \mathbb{A}$$

such that

$$\sigma_1 \pm \sigma_3 \pm \sigma_5 \neq 0 \quad \textit{for all choices of the signs,}$$

there exists a right-angled hexagon $(S_n,\, n \bmod 6)$ *with these sides. It is uniquely determined up to a simultaneous change of the orientations of* S_2, S_4, S_6.

Suppose first that two of the given sides σ_1 and σ_3 say, are different from 0 and πi, thus $\sinh\sigma_1 \sinh\sigma_3 \neq 0$. Then σ_2 must satisfy the equation

$$\cosh\sigma_5 = \cosh\sigma_1 \cosh\sigma_3 + \sinh\sigma_1 \sinh\sigma_3 \cosh\sigma_2\,.$$

Since $\sigma_1, \sigma_3 \in \mathbb{A} \setminus \{0, \pi i\}$, the side-line S_2 must be proper and $\sigma_2 \neq 0, \pi i$. Now, the inequalities assumed secure precisely that $\cosh\sigma_2 \neq \pm 1$ for the solutions. Let σ_2 denote one of the solutions; the other one is then $-\sigma_2$. The quantities $\sigma_1, \sigma_2, \sigma_3$ satisfy the assumptions in case I. Hence, there exists a unique hexagon with these

sides. If its side-line S_5 is suitably oriented, it has the given side σ_5 because of the equation above. The hexagon with the second side $-\sigma_2$ may be obtained by reversing the orientations of S_2, S_4, S_6. This leaves the sides $\sigma_1, \sigma_3, \sigma_5$ unchanged.

Suppose now that two of the given sides, σ_1 and σ_3 say, are 0 or πi. Then S_1 and S_3 have to be improper and σ_6 and σ_4 to be $\pm\infty$. If $\sigma_5 \neq 0, \pi i$, it is no restriction to choose S_5 and any proper oriented normal of it as S_4. Then S_6 is uniquely determined by σ_5, and since S_1 and S_3 have to be improper, S_2 must be a line joining an end of S_4 with an end of S_6. Taking the orientation of S_2 into account, one obtains eight hexagons. In each of the four cases:

$\sigma_1 = 0$ and $\sigma_3 = 0$, $\sigma_1 = 0$ and $\sigma_3 = \pi i$, $\sigma_1 = \pi i$ and $\sigma_3 = 0$, $\sigma_1 = \pi i$ and $\sigma_3 = \pi i$ two of these hexagons satisfy the requirement. A half-turn about S_5 maps however one onto the other with the orientations of S_2, S_4, S_6 reversed. This proves the statement under the present assumptions. If σ_3 also equals 0 or πi, the inequality condition requires that either one or all three of $\sigma_1, \sigma_3, \sigma_5$ are πi. The side-lines S_1, S_3, S_5 have to be improper. Hence, a hexagon in question is a triangle with improper vertices S_1, S_3, S_5 and side-lines S_2, S_4, S_6 suitably oriented. Clearly, the orientations of these lines may be reversed simultaneously. Since any two triangles with improper vertices are congruent, the statement holds also in this case.

The values $\pm\infty$ of $\sigma_1, \sigma_3, \sigma_5$ are admissible but excluded by the assumptions. If one of them is $\pm\infty$, another one must also be $\pm\infty$. Assume for instance that $\sigma_1 = \pm\infty$ and $\sigma_3 = \pm\infty$. Then whatever value of $\sigma_5 \in \mathbb{A}_\infty$ is given, there are infinitely many hexagons satisfying the requirements, even if in addition $\sigma_2 = 0$ or πi is assumed. If $\sigma_5 \neq 0, \pi i$, possibly $\pm\infty$, choose a proper line S_5 and a proper normal S_4 of it. Then S_6, possibly improper, is determined by σ_5. Take any improper S_2 distinct from the ends of S_4, S_5, S_6, and let S_1 and S_3 be the common normals of S_2, S_6 and S_2, S_4, respectively. With suitable orientations of S_1 and S_3 the hexagon obtained satisfies the requirements. If $\sigma_5 = 0$ or πi, any hexagon with S_2 and S_5 improper does.

III. *Given*

$$\sigma_1 \in \mathbb{A}_\infty \setminus \{0, \pi i\}, \quad \sigma_2 \in \mathbb{A} \setminus \{0, \pi i\}, \quad \sigma_4 \in \mathbb{A}_\infty,$$
$$(\sigma_1, \sigma_4) \neq (\pm\infty, \pm\infty), \quad (\sigma_1, \sigma_2) \neq \left(\pm\frac{\pi}{2} i, \pm\frac{\pi}{2} i\right),$$

there exist in general two distinct right-angled hexagons (S_n, n mod 6) *with these sides, in exceptional cases only one or none at all.*

Here several cases have to be distinguished.

Suppose first that σ_1 and σ_4 are finite. Then σ_6 must be finite since otherwise σ_5 should be 0 or πi and thus $\sigma_4 = \pm\infty$, and because of the finiteness of σ_1 it must

be different from 0 and πi. Consequently all side-lines, S_4 possibly excepted, of the hexagons to be determined must be proper, and σ_6 must satisfy the equation

$$\cosh\sigma_2\cosh\sigma_6+\sinh\sigma_2\cosh\sigma_1\sinh\sigma_6=\cosh\sigma_4$$

in which the coefficients of $\cosh\sigma_6$ and $\sinh\sigma_6$ do not vanish simultaneously because of $(\sigma_1,\sigma_2)\neq\left(\pm\frac{\pi}{2}i,\pm\frac{\pi}{2}i\right)$. The discussion of an equation of this type in I.2 shows that it has two distinct solutions provided

$$\cosh^2\sigma_2-\sinh^2\sigma_2\cosh^2\sigma_1\neq 0 \quad\text{and}\quad \neq\cosh^2\sigma_4$$

or, equivalently,

$$\sinh^2\sigma_1\sinh^2\sigma_2\neq 1 \quad\text{and}\quad \neq\sinh^2\sigma_4\,.$$

However, solutions $\sigma_6=0$ and $\sigma_6=\pi i$ have to be rejected. Now, $\sinh\sigma_6=0$ satisfies the equation if and only if

$$\cosh^2\sigma_2=\cosh^2\sigma_4\,,$$

and no other value of $\sinh\sigma_6$ does if $\cosh^2\sigma_2=\cosh^2\sigma_4=0$. Hence, only one or none of the solutions can be used. If

$$\sinh^2\sigma_1\sinh^2\sigma_2=1 \quad\text{or}\quad =\sinh^2\sigma_4\,,$$

the equation has one solution, and this has to be rejected if in addition $\cosh^2\sigma_2=\cosh^2\sigma_4$. Let now $\sigma_6\neq 0,\pi i$ be a solution. Then $\sigma_6,\sigma_1,\sigma_2$ satisfy the assumptions (on $\sigma_1,\sigma_2,\sigma_3$) in I. Hence, there is a hexagon with these sides, and since its sides satisfy the equation above it has the given side σ_4 provided S_4 is suitably oriented.

Suppose now that either $\sigma_1=\pm\infty$ or $\sigma_4=\pm\infty$. The case where $\sigma_1=\pm\infty$ and $\sigma_4=0$ or πi has to be treated separately. Otherwise all side-lines with the exception of either S_6 or S_5 must be proper, and the Law of Cotangents

$$\coth\sigma_1\sinh\sigma_3+\cosh\sigma_2\cosh\sigma_3+\sinh\sigma_2\coth\sigma_4=0$$

must be satisfied. It may be considered as an equation for the unknown σ_3. The conditions for two distinct solutions are

$$\cosh^2\sigma_2-\coth^2\sigma_1\neq 0\,,\quad \cosh^2\sigma_2-\coth^2\sigma_1\neq\sinh^2\sigma_2\coth^2\sigma_4\,.$$

If $\sigma_1=\pm\infty$, hence $\coth^2\sigma_1=1$, both are satisfied since $\sinh\sigma_2\neq 0$, thus $\cosh^2\sigma_2\neq 1$, and $\coth^2\sigma_4\neq 1$. However, a solution such that $\sinh\sigma_3=0$ has to be rejected. This holds for one solution if

$$\coth^2\sigma_2=\coth^2\sigma_4\neq 0$$

and for both if $\coth\sigma_2 = \coth\sigma_4 = 0$. If $\sigma_4 = \pm\infty$, hence $\coth^2\sigma_4 = 1$, the second of the inequalities above holds, so there are two or one solution according as the first inequality does or does not hold. There are no solutions with $\sinh\sigma_3 = 0$, since $\cosh^2\sigma_2 \neq \sinh^2\sigma_2$. If σ_3 is an admissible solution, $\sigma_1, \sigma_2, \sigma_3$ satisfy the assumptions under I and one sees as before that there is one and only one hexagon with the sides $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

It remains to consider the case $\sigma_1 = \pm\infty$, $\sigma_4 = 0$ or πi. For each of the four pairs (σ_1, σ_4) of these values and the given $\sigma_2 \in \mathbb{A}\setminus\{0, \pi i\}$ there are two hexagons with these sides. They may be constructed directly. Choose S_1 and a normal S_2 of it, both proper. The necessarily improper side-line S_6 must coincide with an end of S_1, which of the ends depends on the sign of σ_1. Further, S_3 is determined by S_1, S_2 and the value of σ_2, and S_4 has to be improper and thus one of the ends of S_3. This may by chosen arbitrarily, and S_5 must then be the line joining S_4 and S_6. The orientation of S_5 has to be chosen such that σ_4 obtains the given value 0 or πi.

Also here certain admissible values of the given sides are excluded by the assumptions under III, the reason being again that, if they are given, infinitely many hexagons with these sides exist. The following cases have to be considered:

$$1)\ \sigma_1 = \pm\frac{\pi}{2}i,\quad \sigma_2 = \pm\frac{\pi}{2}i, \qquad 2)\ \sigma_1 = 0 \quad\text{or}\quad \pi i,$$

$$3)\ \sigma_2 = 0 \quad\text{or}\quad \pi i, \qquad 4)\ \sigma_2 = \pm\infty\,.$$

1) Choose proper lines S_1 and S_2 intersecting orthogonally. Then S_6 and S_3 must coincide with their common normal. Since S_4 and S_5 must intersect each other and the coincident lines S_3 and S_6 orthogonally, we have $\sigma_4 = \pm\frac{\pi}{2}i$ $\left(\text{and } \sigma_5 = \pm\frac{\pi}{2}i\right)$. Choosing the orientations appropriately a hexagon with the given sides $\sigma_1, \sigma_2, \sigma_4$ and any $\sigma_3 \in \mathbb{A}\setminus\{0, \pi i\}$ can be obtained.

2) S_1 is improper by assumption and consequently $\sigma_2 = \pm\infty$. Choose S_1 and a line S_2 ending at S_1, further a proper normal S_3 of S_2. If the given value σ_4 is 0 or πi, let S_4 be one of the ends of S_3. Choose any line S_5 ending at S_4 but not at S_1, distinct from S_3, and oriented suitably. A hexagon with the prescribed sides is obtained by letting S_6 be the normal of S_5 ending at S_1. If $\sigma_4 \in \mathbb{A}\setminus\{0, \pi i\}$, choose any $\sigma_3 \in \mathbb{A}\setminus\{0, \pi i\}$. Then S_4 is determined when S_2 and S_3 have been chosen. Now σ_4 determines S_5, possibly improper, and with S_6 the common normal of S_1 and S_5 a hexagon with the prescribed sides is obtained. (There are values of σ_3 for which S_5 ends at or coincides with S_1. They have to be excluded.)

3) S_2 is improper by assumption and consequently $\sigma_1 = \pm\infty$ and $\sigma_3 = \pm\infty$. Choose S_2 and a proper S_3 ending at S_2. If $\sigma_4 = 0$ or πi, S_4 has to be the other end of S_3. Any choice of S_1 ending at S_2 and of S_5 ending at S_4 yields a hexagon satisfying the requirements. If $\sigma_4 \neq 0, \pi i$, let S_4 be any proper line normal to S_3.

Then S_5, possibly improper, is determined by σ_4. For any proper S_1 ending at S_2 a hexagon with the given sides is obtained.
4) The cases $\sigma_1 = 0$ or πi are covered by 2). Otherwise S_3 must be improper, and this requires $\sigma_4 = \pm\infty$. Choose a proper line S_1 and a proper normal S_2 of it. Then S_6, possibly improper, is determined by σ_1, and S_3 must be one of the ends of S_2. Choose any proper line S_4 ending at S_3. The hexagon with S_5 the common normal of S_4 and S_6 satisfies the requirement.

In the following we discuss briefly some of the special cases concerning plane polygons of the problems dealt with above. However, a more direct approach appears to be simpler than reference to the general results. We use the notations introduced in VI.3 and refer to the figures there. The use of the positive side-lengths and the interior angles instead of the complex sides implies loss of information, for instance, whether an end point of a side is the vertex of a right or non-right angle. Therefore, there may be polygons of different types having the same three given lengths or angles.

Suppose, for instance, that lengths a, c', b of three consecutive sides are given. Start by choosing a segment AB of length c'. Orthogonal to AB lay off a segment AA' of length a and a segment BB' of length b to the same side or to opposite sides of AB. Clearly the quadrangles $ABB'A'$ with two adjacent right angles satisfy the requirements. Drawing the line through A' orthogonal to AA' and then the normal of this line through B', one obtains a pentagon, convex or self-intersecting with four right angles, which has the given side-lengths. Another such pentagon is obtained by interchanging the roles of A, A' and B, B'. Further, draw both the line through A' orthogonal to AA' and the line through B' orthogonal to BB'. If these lines intersect or are parallel, one gets another pentagon with four right angles. If the lines are ultraparallel, their common normal is the sixth side of a right-angled hexagon, convex or self-intersecting, with the given side-lengths.
We shall not discuss all these cases in detail but only show:

Positive numbers a, c', b are the lengths of consecutive sides of a convex right-angled hexagon if and only if

$$\cosh c' > \frac{1 + \cosh a \cosh b}{\sinh a \sinh b}.$$

The hexagon is unique up to isometries of the plane.

The necessity of the condition follows from the relation

$$(*) \qquad \cosh c = -\cosh a \cosh b + \sinh a \sinh b \cosh c'$$

valid for a right-angled hexagon because $\cosh c > 1$ for the side c opposite c'. To see the sufficiency observe that if the construction described above with A' and B'

on the same side of AB and right angles at A' and B' does not lead to a right-angled hexagon, it gives a pentagon with four right angles and a certain angle γ opposite c' or a self-intersecting hexagon. In the first case $\cos\gamma$ or $-\cos\gamma$ (according as the pentagon is convex or self-intersecting) equals the right-hand side of $(*)$. In the second case the right-hand side equals $-\cosh c$, for c' and its opposite side c do not intersect, since A' and B' lie on the same side of the line AB. Hence, in both cases the right-hand side of $(*)$ is less than or equal to 1 and, thus, the condition not satisfied. The construction shows the uniqueness.

Given any three positive numbers a, b, c, there is a unique convex right-angled hexagon with alternating sides of lengths a, b, c.

To see this, observe that there is one and only one positive number c' such that

$$\cosh c' = \frac{\cosh c + \cosh a \cosh b}{\sinh a \sinh b} > \frac{1 + \cosh a \cosh b}{\sinh a \sinh b}.$$

According to the previous result there is a unique convex right-angled hexagon with the lengths a, c', b of consecutive sides. The equation defining c' shows that the side opposite c' has the length c.

We mention, omitting the simple proof: There is in addition a self-intersecting right-angled hexagon with alternative sides of lengths a, b, c if and only if one of these numbers is greater than the sum of the two others.

In the following the questions of the type considered are discussed for triangles. Uniqueness is always to be understood up to isometries of the plane.

We start by stating the obvious fact:

Given numbers a, b, γ satisfying $0 < a \leq -\infty, 0 < b \leq \infty, 0 < \gamma < \pi$, there is one and only one triangle with two sides of lengths a and b and the angle γ between them.

Given three positive numbers a, b, c satisfying the triangle inequalities $|a - b| < c < a + b$, there is one and only one triangle with sides of lengths a, b, c.

To see this, choose a segment AB of length c. The triangle inequalities imply that the circle with centre A and radius b intersects the circle with centre B and radius a. Each of the intersection points determines together with A und B a triangle satisfying the requirements, and these two triangles are congruent. Obviously there are no others.

Given positive numbers α, β, c such that $\alpha + \beta < \pi$, there is one and only one triangle with a side of length c and the adjacent angles α and β if and only if

$$(*) \qquad \cosh c \leq \frac{1 + \cos\alpha \cos\beta}{\sin\alpha \sin\beta}.$$

That $(*)$ is necessary follows from the fact that the third angle of a triangle with properties required satisfies

$$-\cos\alpha\cos\beta+\sin\alpha\sin\beta\cosh c=\cos\gamma\leq 1\,.$$

To see that $(*)$ is sufficient, choose a segment AB of length c. Let H denote one of the half-planes bounded by the line AB and H' the other one. Draw the line through A which in H forms the angle α with the segment AB and the line through B which in H forms the angle β with BA. The statement is then that these lines intersect or are parallel in H if $(*)$ is satisfied. Clearly, they cannot intersect or be parallel in H' since one then would have a triangle with angle sum greater than π. Suppose that the lines were ultraparallel. If the joining segment of their common normal were contained in H', there would be a convex quadrangle with two adjacent right angles and the sum of the other angles greater than π. If this segment were contained entirely or partly in H, it would be the side joining the vertices of the right angles of a convex or self-intersecting quadrangle with two adjacent right angles. The other angles would be α, β in the convex case and α, $\pi-\beta$ or $\pi-\alpha$, β in the other one. If c' denotes the length of the segment, we would have in both cases (cf. VI.3, 3), 3'))

$$-\cos\alpha\cos\beta+\sin\alpha\sin\beta\cosh c=\cosh c'>1$$

which is impossible because of $(*)$.

Given non-negative numbers α,β,γ such that

$$\alpha+\beta+\gamma<\pi\,,$$

there is one and only one triangle with angles α,β,γ.

Assume first that at least two of the given numbers, say α and β, are positive. Determine $c>0$ by

$$\cosh c=\frac{\cos\gamma+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}\,.$$

This is possible since

$$\frac{\cos\gamma+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-1=\frac{\cos\gamma+\cos(\alpha+\beta)}{\sin\alpha\sin\beta}>0$$

because of $\cos(\pi-\gamma)<\cos(\alpha+\beta)$. Now α,β,c obviously satisfy $(*)$. Hence, there exists a unique triangle with a side of length c and the adjacent angles α and β. This triangle and no other satisfies the requirements.

If two of the given numbers, say β and γ, are zero, the statement is obviously true. Choose two half-lines issuing from the same proper point and forming an

angle of size α. The triangle in question is then obtained by joining the ends of the half-lines.

The triangles with all vertices improper, and only these have angles $\alpha = \beta = \gamma = 0$, and they are mutually congruent.

Given numbers $\alpha > 0, \beta > 0, b > 0$ such that $\alpha + \beta < \pi$, there is one and only one triangle with a side of length b, an adjacent angle α and the opposite angle β.

The third angle γ must satisfy

$$\cos\beta = -\cos\alpha\cos\gamma + \sin\alpha\sin\gamma\cosh b\,.$$

Consider

$$\frac{\cos\beta + \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma} = \frac{\cos\beta + \cos(\alpha+\gamma)}{\sin\alpha\sin\gamma} + 1$$

as a function of γ in the interval $0 < \gamma < \pi - (\alpha + \beta)$. It is continuous and strictly decreasing from $+\infty$ to 1. Hence, there is one and only one γ in the interior of the interval for which the function has the value $\cosh b$. The triangle with the angles α, β and this γ, and no other satisfies the requirements.

Given positive numbers $a, b, \beta < \gamma$, there is a triangle with sides of lengths a and b and the angle β opposite the latter if and only if these numbers satisfy either $a < b$ or $a = b,\ \beta < \frac{\pi}{2}$ or $a > b,\ \beta < \frac{\pi}{2}$ and

$$(**)\qquad \sinh a \sin\beta \le \sin b\,.$$

There is only one such triangle unless $a > b, \beta < \frac{\pi}{2}$ and $<$ is valid in $(**)$ when there are two.

To see this, choose a segment BC of length a and draw one of the half-lines from B forming the angle β with BC. A point A is the third vertex of a triangle of the kind required if and only if it lies on the half-line and on the circle with centre C and radius b. If $a < b$ or $a = b, \beta < \frac{\pi}{2}$, the half-line obviously intersects the circle once. If $a > b$, the angle β has to be acute and the distance d from c to the half-line not greater than b to have intersection. Since $\sinh d = \sinh a \sin\beta$, this is equivalent to $(**)$. There are two distinct intersection points if strict inequality holds.

VI.5 The amplitudes of a right-angled hexagon

We return to the study of right-angled hexagons in space. With such a hexagon $(S_n, n \bmod 6)$ we associate two complex numbers which are generalizations of the sine amplitude of spherical trigonometry. They will be called the *amplitudes* of the hexagon and denoted by $am(S_1, S_3, S_5)$ and $am(S_2, S_4, S_6)$ since each of them only depends on the respective side-lines.

If the side-lines S_1, S_3, S_5 are proper and $\mathbf{s}_1, \mathbf{s}_3, \mathbf{s}_5$ normalized line matrices determining them, then by definition

$$am(S_1, S_3, S_5) = -\frac{1}{2}\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1).$$

If at least one of S_1, S_3, S_5 is improper,

$$am(S_1, S_3, S_5) = \infty .$$

Analogously $am(S_2, S_4, S_6)$ is defined. Obviously, both are invariant under motions.

Of two statements or relations obtainable from each other by interchange of the odd and even subscripts only one is given in the sequel.

By definition of the hexagon (cf. VI.1), S_1, S_3, S_5, now assumed to be proper, have no common normal. Hence (cf. V.1)

$$am(S_1, S_3, S_5) \neq 0 .$$

Clearly, $am(S_1, S_3, S_5)$ changes sign if the orientation of one of S_1, S_3, S_5 is reversed. Further, $am(S_1, S_3, S_5)$ remains unchanged under even (cyclic) permutations of S_1, S_3, S_5. It changes sign under odd permutations, indeed

$$\begin{aligned}\operatorname{tr}(\mathbf{s}_5\mathbf{s}_1\mathbf{s}_3) &= \operatorname{tr}(\mathbf{s}_5\mathbf{s}_1\mathbf{s}_3)^{\sim} = \operatorname{tr}(\tilde{\mathbf{s}}_3\tilde{\mathbf{s}}_1\tilde{\mathbf{s}}_5)\\ &= -\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5) = -\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1).\end{aligned}$$

Let s_1, s_3, s_5 denote the half-turns about the lines S_1, S_3, S_5 and $\delta_{od} = \delta(s_5 \circ s_3 \circ s_1)$ the displacement, determined up to sign, of the motion $s_5 \circ s_3 \circ s_1$. From IV.2(1) we then obtain

$$am^2(S_1, S_3, S_5) = \frac{1}{2}(\cosh\delta_{od} + 1) = \cosh^2\frac{\delta_{od}}{2}. \tag{1}$$

The expressions for $am(S_1, S_3, S_5)$ to be derived fail if all of S_2, S_4, S_6 are improper, that is, for a triangle with improper vertices. Therefore we compute $am(S_1, S_3, S_5)$ directly in this case. Let S_1, S_3, S_5 be oriented in accordance with this cyclical order. We may then assume that $S_1 = [-1, 1]$, $S_3 = [1, \infty]$, $S_5 = [\infty, -1]$. The corresponding line matrices are (cf. V.2)

$$\mathbf{s}_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{s}_3 = \begin{pmatrix} i & -2i \\ 0 & -i \end{pmatrix}, \quad \mathbf{s}_5 = \begin{pmatrix} -i & -2i \\ 0 & i \end{pmatrix}$$

for which $\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1) = -4i$. Consequently we have in this case

$$am(S_1, S_3, S_5) = 2i\,.$$

Assume now that at least one of S_2, S_4, S_6, say S_4, is proper. Then

$$\begin{aligned}\operatorname{tr}(\mathbf{s}_5\mathbf{s}_4\mathbf{s}_3)\operatorname{tr}(\mathbf{s}_4\mathbf{s}_1) &= \operatorname{tr}(\mathbf{s}_5\mathbf{s}_4\mathbf{s}_3\mathbf{s}_4\mathbf{s}_1) + \operatorname{tr}(\mathbf{s}_5\mathbf{s}_4\mathbf{s}_3\mathbf{s}_1\mathbf{s}_4) \\ &= 2\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)\,.\end{aligned}$$

This yields (cf. V.3(2))

$$am(S_1, S_3, S_5) = i \sinh\sigma_4 \cosh\alpha_{14}\,, \tag{2}$$

where α_{14} denotes the altitude introduced in VI.2. If S_2 is also proper, we have as shown in VI.2,

$$\cosh\alpha_{14} = -\sinh\sigma_2\sinh\sigma_3$$

and thus

$$am(S_1, S_3, S_5) = -i\sinh\sigma_2\sinh\sigma_3\sinh\sigma_4\,. \tag{3}$$

If all side-lines are proper, we also have

$$\begin{aligned}am(S_1, S_3, S_5) &= -i\sinh\sigma_4\sinh\sigma_5\sinh\sigma_6 \\ &= -i\sinh\sigma_6\sinh\sigma_1\sinh\sigma_2\,.\end{aligned}$$

Using the analogue expressions for $am(S_2, S_4, S_6)$, we may write the Law of Sines in the form

$$\frac{\sinh\sigma_1}{\sinh\sigma_4} = \frac{\sinh\sigma_3}{\sinh\sigma_6} = \frac{\sinh\sigma_5}{\sinh\sigma_2} = \frac{am(S_2, S_4, S_6)}{am(S_1, S_3, S_5)}\,. \tag{4}$$

Further,

$$am^2(S_2, S_4, S_6) = -\sinh\sigma_1\sinh\sigma_2\sinh^2\sigma_3\sinh\sigma_4\sinh\sigma_5$$

yields

$$am(S_1, S_3, S_5) = \frac{i\,am^2(S_2, S_4, S_6)}{\sinh\sigma_1\sinh\sigma_3\sinh\sigma_5}\,. \tag{5}$$

Assume again that S_1, S_3, S_5 are proper while S_2, S_4, S_6 may be improper. Then the matrices

$$\mathbf{a} = -\mathbf{s}_5\mathbf{s}_3\,, \quad \mathbf{b} = -\mathbf{s}_1\mathbf{s}_5\,, \quad \mathbf{c} = -\mathbf{s}_3\mathbf{s}_1$$

have determinants equal to 1 and $\mathbf{c\,b\,a} = \mathbf{1}$ because $\mathbf{s}_1^2 = \mathbf{s}_3^2 = \mathbf{s}_5^2 = -\mathbf{1}$. Since

$$\operatorname{tr}(\mathbf{a\,b\,c}) = -\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)^2 = 2 - \operatorname{tr}^2(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)\,,$$

I.3(6) yields

$$\operatorname{tr}^2(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1) = \frac{1}{2}\begin{vmatrix} 2 & -\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3) & -\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) \\ -\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3) & 2 & -\operatorname{tr}(\mathbf{s}_1\mathbf{s}_5) \\ -\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) & -\operatorname{tr}(\mathbf{s}_1\mathbf{s}_5) & 2 \end{vmatrix}$$

and thus (cf. I.2(1))

$$\begin{aligned} am^2(S_1, S_3, S_5) &= \begin{vmatrix} 1 & \cosh\sigma_4 & \cosh\sigma_2 \\ \cosh\sigma_4 & 1 & \cosh\sigma_6 \\ \cosh\sigma_2 & \cosh\sigma_6 & 1 \end{vmatrix} \\ &= 1 - \cosh^2\sigma_2 - \cosh^2\sigma_4 - \cosh^2\sigma_6 + 2\cosh\sigma_2\cosh\sigma_4\cosh\sigma_6 \\ &= 4\sinh\sigma_{ev}\sinh(\sigma_{ev}-\sigma_2)\sinh(\sigma_{ev}-\sigma_4)\sinh(\sigma_{ev}-\sigma_6)\,, \end{aligned} \tag{6}$$

where

$$\sigma_{ev} = \frac{1}{2}(\sigma_2 + \sigma_4 + \sigma_6)$$

with arbitrarily chosen values of $\frac{1}{2}\sigma_2, \frac{1}{2}\sigma_4, \frac{1}{2}\sigma_6$.

Applying I.2(2), we obtain

$$\begin{aligned} am^2(S_1, S_3, S_5) = 16\sinh^2\frac{\sigma_2}{2}\sinh^2\frac{\sigma_4}{2}\sinh^2\frac{\sigma_6}{2} \\ + 4\left(\sinh\frac{\sigma_2}{2} + \sinh\frac{\sigma_4}{2} + \sinh\frac{\sigma_6}{2}\right) \\ \cdot\left(-\sinh\frac{\sigma_2}{2} + \sinh\frac{\sigma_4}{2} + \sinh\frac{\sigma_6}{2}\right) \\ \cdot\left(\sinh\frac{\sigma_2}{2} - \sinh\frac{\sigma_4}{2} + \sinh\frac{\sigma_6}{2}\right) \\ \cdot\left(\sinh\frac{\sigma_2}{2} + \sinh\frac{\sigma_4}{2} - \sinh\frac{\sigma_6}{2}\right) \end{aligned} \tag{7}$$

which will be used later on.

With (6) for $am^2(S_2, S_4, S_6)$, (5) gives an expression for $am(S_1, S_3, S_5)$ in terms of the odd-numbered sides.

From (3) one obtains

$$am^2(S_1, S_3, S_5)(\sinh^2\sigma_1 + \sinh^2\sigma_3 + \sinh^2\sigma_5) \\ = am^2(S_2, S_4, S_6)(\sinh^2\sigma_2 + \sinh^2\sigma_4 + \sinh^2\sigma_6)\,.$$

Using (6) written

$$\sinh^2\sigma_2 + \sinh^2\sigma_4 + \sinh^2\sigma_6 \\ = 2(\cosh\sigma_2\cosh\sigma_4\cosh\sigma_6 - 1) - am^2(S_1, S_3, S_5)$$

and the analogue with the odd and even subscripts interchanged one sees that

$$\frac{am^2(S_1, S_3, S_5)}{am^2(S_2, S_4, S_6)} = \frac{\cosh\sigma_2\cosh\sigma_4\cosh\sigma_6 - 1}{\cosh\sigma_1\cosh\sigma_3\cosh\sigma_5 - 1}\,.$$

Further relations in which the amplitudes are involved can be obtained by means of the Law of Cosines. Multiplying VI.2(6) for $n = 2, 4, 6$, dividing by $\sinh\sigma_1\sinh\sigma_3\sinh\sigma_5$, and using (6), (3) and (5), we obtain

$$\text{(8)}\qquad \frac{(\cosh\sigma_1 - 1)(\cosh\sigma_3 - 1)(\cosh\sigma_5 - 1)}{am^2(S_2, S_4, S_6)} = \frac{2\sinh^2\sigma_{ev}}{am^2(S_1, S_3, S_5)}\,,$$

which will be used later on. Elimination by means of (5) of $am(S_2, S_4, S_6)$ yields

$$\text{(9)}\qquad \tanh\frac{\sigma_1}{2}\tanh\frac{\sigma_3}{2}\tanh\frac{\sigma_5}{2} = -\frac{2i\sinh^2\sigma_{ev}}{am(S_1, S_3, S_5)}$$

and of $am(S_1, S_3, S_5)$

$$\text{(10)}\qquad \sinh^2\sigma_{ev} = -\frac{am^2(S_2, S_4, S_6)}{16\cosh^2\frac{\sigma_1}{2}\cosh^2\frac{\sigma_3}{2}\cosh^2\frac{\sigma_5}{2}}\,.$$

We specialize some of these formulae to triangles.

As in VI.3, we consider a triangle with side-lengths a, b, c and angles α, β, γ as a hexagon with sides

$$\sigma_1 = a\,,\qquad \sigma_3 = b\,,\qquad \sigma_5 = c\,,$$
$$\sigma_4 = (\pi - \alpha)i\,,\quad \sigma_6 = (\pi - \beta)i\,,\quad \sigma_2 = (\pi - \gamma)i\,.$$

Instead of $am(S_1, S_3, S_5)$ and $am(S_2, S_4, S_6)$ we write am_s and am_v, s and v standing for "side-lines" and "vertices", respectively. Most of the following relations are only valid if all angles are positive and, thus, the lengths of the sides finite. In cases where zero angles are admitted, this will be indicated.

The length h_a of the altitude from the vertex of α is related to the altitude α_{14} of the hexagon by $\alpha_{14} = \pm(h_a + \frac{\pi}{2}i)$. Hence we obtain from (2)

$$(11)\qquad \begin{aligned} am_s &= i \sin\alpha \sinh h_a, \qquad \beta \geq 0, \gamma \geq 0, \\ am_v &= \sinh a \sinh h_a, \end{aligned}$$

and the analogues; (3) yields

$$(12)\qquad \begin{aligned} am_s &= i \sin\alpha \sinh b \sinh\gamma, \qquad \beta \geq 0, \\ am_v &= \sinh a \sinh\beta \sinh c \end{aligned}$$

and the analogues. We note that

$$-i\,am_s > 0, \qquad am_v > 0.$$

Specialization of (6) yields for $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$

$$(13)\qquad \begin{aligned} am_s^2 &= 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma \\ &= -4\cos\frac{1}{2}(\alpha+\beta+\gamma)\cos\frac{1}{2}(-\alpha+\beta+\gamma)\cos\frac{1}{2}(\alpha-\beta+\gamma) \\ &\qquad \cos\frac{1}{2}(\alpha+\beta-\gamma). \end{aligned}$$

Since the right-hand side is greater than or equal to -4, we have

$$0 < -i\,am_s \leq 2$$

with equality only for $\alpha = \beta = \gamma = 0$, that is, for the triangle with all vertices improper. For am_v (6) gives

$$(14)\qquad \begin{aligned} am_v^2 &= 1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a\cosh b\cosh c \\ &= 4\sinh\frac{1}{2}(a+b+c)\sinh\frac{1}{2}(-a+b+c)\sinh\frac{1}{2}(a-b+c) \\ &\qquad \cdot\sinh\frac{1}{2}(a+b-c). \end{aligned}$$

Expressions for the perimeter in terms of the angles and for the sum of the angles in terms of the sides are obtained from (10). They may be written

$$(15)\qquad \sinh\frac{1}{2}(a+b+c) = \frac{-i\,am_s}{4\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma},$$

$$(16)\qquad \cos\frac{1}{2}(\alpha+\beta+\gamma) = \frac{am_v}{4\cosh\frac{1}{2}a\cosh\frac{1}{2}b\cosh\frac{1}{2}c}.$$

The last relation shows that

$$(17)\qquad \cos\frac{1}{2}(\alpha+\beta+\gamma) < \frac{1}{4}am_v.$$

For later use we specialize also to the *spherical triangle*. Herewe have to put (cf. VI.3)

$$\sigma_1 = ai\,, \qquad \sigma_3 = bi\,, \qquad \sigma_5 = ci\,,$$
$$\sigma_4 = (\pi - \alpha)\,i\,, \quad \sigma_6 = (\pi - \beta)\,i\,, \quad \pi_2 = (\pi - \gamma)\,i\,.$$

With obvious interpretations of am_s and am_v we obtain

$$-am_s = \sin\alpha\sin\beta\sin c = \sin a\sin\beta\sin\gamma = \sin\alpha\sin b\sin\gamma\,, \tag{18}$$

$$-am_v = \sin a\sin b\sin\gamma = \sin\alpha\sin b\sin c = \sin a\sin\beta\sin c\,, \tag{19}$$

$$\begin{aligned} am_s^2 &= 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma \\ &= -4\cos\frac{1}{2}(\alpha+\beta+\gamma)\cos\frac{1}{2}(-\alpha+\beta+\gamma) \\ &\qquad\cdot\cos\frac{1}{2}(\alpha-\beta+\gamma)\cos\frac{1}{2}(\alpha+\beta-\gamma)\,, \end{aligned} \tag{20}$$

$$\begin{aligned} am_v^2 &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c \\ &= 4\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(-a+b+c)\sin\frac{1}{2}(a-b+c) \\ &\qquad\cdot\sin\frac{1}{2}(a+b-c)\,, \end{aligned} \tag{21}$$

$$\sin^2\frac{1}{2}(a+b+c) = \frac{am_s^2}{2(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)}\,, \tag{22}$$

$$\cos^2\frac{1}{2}(\alpha+\beta+\gamma) = \frac{am_v^2}{2(1+\cos a)(1+\cos b)(1+\cos c)}\,. \tag{23}$$

For spherical triangles with all sides and angles positive and less than π we have evidently

$$-1 \le am_s < 0\,, \qquad -1 \le am_v < 0\,. \tag{24}$$

VI.6 Transversals of a right-angled hexagon

We consider again an arbitrary right-angled hexagon (S_n, $n \bmod 6$). Let $\mathbf{s}_n$ denote any line matrices determining the side-lines S_n. We observe that the assumption, that S_{n-1} and S_{n+1} do not coincide, implies that two opposite side-lines S_n and S_{n+3} cannot be normals of each other, hence that

(1) $\quad \operatorname{tr}(\mathbf{s}_{n+3}\mathbf{s}_n) \neq 0 .$

Indeed, if, for instance, S_4 were normal to S_1, then S_2 would be a normal of S_1 distinct from S_4, and consequently S_3 would coincide with S_1.

Any line T_n normal to S_n is a transversal of the double cross $(S_{n-1}, S_{n+1}; S_n)$ as defined in V.4. It will be called a *transversal* of the hexagon. Its *co-transversal* T_n^* is by definition the common normal of T_n and the side-line S_{n+3} opposite S_n. A line matrix for T_n may be written

$$\mathbf{t}_n = p_n \mathbf{s}_{n-1} + q_n \mathbf{s}_{n+1}, \quad p_n, q_n \in \mathbb{C}, \quad (p_n, q_n) \neq (0,0),$$

and a line matrix for T_n^*

$$\mathbf{t}_n^* = p_n^* \mathbf{s}_{n+2} + q_n^* \mathbf{s}_{n-2}, \quad p_n^*, q_n^* \in \mathbb{C}, \quad (p_n^*, q_n^*) \neq (0,0).$$

Since T_n^* is normal to T_n, we must have

$$\begin{aligned} &\operatorname{tr}[(p_n^* \mathbf{s}_{n+2} + q_n^* \mathbf{s}_{n-2})(p_n \mathbf{s}_{n-1} + q_n \mathbf{s}_{n+1})] \\ &\qquad = p_n^* p_n \operatorname{tr}(\mathbf{s}_{n+2}\mathbf{s}_{n-1}) + q_n^* q_n \operatorname{tr}(\mathbf{s}_{n-2}\mathbf{s}_{n+1}) = 0 . \end{aligned}$$

The two traces occurring here do not vanish, so p_n^*, q_n^* must be proportional to $-q_n \operatorname{tr}(\mathbf{s}_{n-2}\mathbf{s}_{n+1})$, $p_n \operatorname{tr}(\mathbf{s}_{n+2}\mathbf{s}_{n-1})$. Hence we may choose

(2) $\quad \mathbf{t}_n^* = -q_n \operatorname{tr}(\mathbf{s}_{n-2}\mathbf{s}_{n+1})\, \mathbf{s}_{n+2} + p_n \operatorname{tr}(\mathbf{s}_{n+2}\mathbf{s}_{n-1})\mathbf{s}_{n-2} .$

A simple consequence is the following generalization of the Menelaos and Ceva theorems (where the odd and even subscripts, of course, may be interchanged):

Transversals T_1, T_3, T_5, determined by the matrices

$$\mathbf{t}_1 = p_1 \mathbf{s}_6 + q_1 \mathbf{s}_2, \quad \mathbf{t}_3 = p_3 \mathbf{s}_2 + q_3 \mathbf{s}_4, \quad \mathbf{t}_5 = p_5 \mathbf{s}_4 + q_5 \mathbf{s}_6,$$

of a right-angled hexagon $(S_n, n \bmod 6)$ have a common normal if and only if

$$p_1 p_3 p_5 + q_1 q_3 q_5 = 0 .$$

Their co-transversal T_1^*, T_3^*, T_5^* have a common normal if and only if

$$p_1 p_3 p_5 - q_1 q_3 q_5 = 0 .$$

These common normals are unique.

A necessary and sufficient condition for T_1, T_3, T_5 to have a common normal is

$$\begin{aligned} &\operatorname{tr}[(p_1 \mathbf{s}_6 + q_1 \mathbf{s}_2)(p_3 \mathbf{s}_2 + q_3 \mathbf{s}_4)(p_5 \mathbf{s}_4 + q_5 \mathbf{s}_6)] \\ &\qquad = p_1 p_3 p_5 \operatorname{tr}(\mathbf{s}_6 \mathbf{s}_2 \mathbf{s}_4) + q_1 q_3 q_5 \operatorname{tr}(\mathbf{s}_2 \mathbf{s}_4 \mathbf{s}_6) \\ &\qquad = (p_1 p_3 p_5 + q_1 q_3 q_5) \operatorname{tr}(\mathbf{s}_2 \mathbf{s}_4 \mathbf{s}_6) = 0 , \end{aligned}$$

and this is the first statement. Applying it to the coefficients

$$\begin{aligned} p_1^* &= -q_1 \operatorname{tr}(\mathbf{s}_5 \mathbf{s}_2), & q_1^* &= p_1 \operatorname{tr}(\mathbf{s}_3 \mathbf{s}_6), \\ p_3^* &= -q_3 \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_4), & q_3^* &= p_3 \operatorname{tr}(\mathbf{s}_5 \mathbf{s}_2), \\ p_5^* &= -q_5 \operatorname{tr}(\mathbf{s}_3 \mathbf{s}_6), & q_5^* &= p_5 \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_4) \end{aligned}$$

in the expressions for matrices $\mathbf{t}_1^*, \mathbf{t}_3^*, \mathbf{t}_5^*$ determining T_1^*, T_3^*, T_5^* one obtains the second statement. Neither T_1, T_3, T_5 or T_1^*, T_3^*, T_5^* can coincide since $\mathbf{s}_2, \mathbf{s}_4, \mathbf{s}_6$ as well as $\mathbf{s}_1, \mathbf{s}_3, \mathbf{s}_5$ are linearly independent. This shows the uniqueness of the common normals.

If S_2, S_4, S_6 are proper and $\mathbf{s}_n$ normalized for the proper S_n,

$$r_1 = -q_1/p_1, \quad r_3 = -q_3/p_3, \quad r_5 = -q_5/p_5$$

admit of a geometrical interpretation. As shown in V.4 we have for $n = 1, 3, 5$

$$\begin{aligned} r_n &= \frac{\sinh \mu(S_{n-1}, T_n; S_n)}{\sinh \mu(S_{n+1}, T_n; S_n)} \\ &= \frac{\sinh[\mu(S_{n-1}, S_{n+1}; S_n) + \mu(S_{n+1}, T_n; S_n)]}{\sinh \mu(S_{n+1}, T_n; S_n)} \\ &= \cosh \sigma_n + \sinh \sigma_n \coth \mu(S_{n+1}, T_n; S_n) \end{aligned}$$

provided the lines involved are proper, the last expression being valid also if T_n is improper.

The Menelaos and Ceva theorems may therefore be formulated as follows.

Proper transversals T_1, T_3, T_5 of a right-angled hexagon with proper side-lines S_n have a common normal if and only if

$$\frac{\sinh \mu(S_6, T_1; S_1) \sinh \mu(S_2, T_3; S_3) \sinh \mu(S_4, T_5; S_5)}{\sinh \mu(S_2, T_1; S_1) \sinh \mu(S_4, T_3; S_3) \sinh \mu(S_6, T_5; S_5)} = 1$$

The co-transversals T_1^*, T_3^*, T_5^* of T_1, T_3, T_5 have a common normal if and only if

$$\frac{\sinh \mu(S_6, T_1; S_1) \sinh \mu(S_2, T_3; S_3) \sinh \mu(S_4, T_5; S_5)}{\sinh \mu(S_2, T_1; S_1) \sinh \mu(S_4, T_3; S_3) \sinh \mu(S_6, T_5; S_5)} = -1$$

We specialize to triangles, using the notations introduced in VI.3 Distances on the side-lines and angles at the vertices are taken with signs in accordance with the orientations chosen there. Transversals of interest here either lie in the plane of the triangle or are orthogonal to it. Let A, B, C as usual denote the vertices of a triangle and P, Q, R points on the lines BC, CA, AB, respectively. We assume all of these points to be proper.

Consider first transversals through P, Q, R orthogonal to the plane of the triangle. The theorem above then gives:

Points P, Q, R on the lines BC, CA, AB, respectively, are collinear if and only if

$$\sinh BP \sinh CQ \sinh AR = \sinh CP \sinh AQ \sinh BR .$$

The co-transversals of transversals of the type considered are the lines AP, BQ, CR. Hence we obtain:

The lines AP, BQ, CR have a common normal, that is, they have a proper or improper point in common or are ultraparallel with a common normal, if and only if

$$\sinh BP \sinh CQ \sinh AR = -\sinh CP \sinh AQ \sinh BR ,$$

For transversals in the plane of the triangle orthogonal to the sides of the latter we obtain:

Three lines in the plane of a triangle ABC which intersect the lines BC, CA, AB orthogonally at P, Q, R, respectively, have a common normal if and only if

$$\cosh BP \cosh CQ \cosh AR = \cosh CP \cosh AQ \cosh BR .$$

To see this one has only to observe that here, for instance,

$$\mu(S_6, T_1; S_1) = BC + \frac{1}{2}\pi i .$$

Finally we consider transversals in the plane of the triangle ABC passing through its vertices. It is convenient to introduce here notations deviating somewhat from those used so far. Let T_4 be a line through A oriented arbitrarily. Denote by α' the angle from the side AC, oriented from A towards C, to T_4 and by α'' the angle from the side AB, oriented from A towards B, to T_4. Analogously β', β'' and γ', γ'' are defined for lines T_6 and T_2 through B and C, respectively. Then we have

$$\begin{aligned}
\mu(S_3, T_4; S_4) &= (\alpha' + \pi)i , & \mu(S_5, T_4; S_4) &= \alpha'' i ,\\
\mu(S_5, T_6; S_6) &= (\beta' + \pi)i , & \mu(S_1, T_6; S_6) &= \beta'' i ,\\
\mu(S_1, T_2; S_2) &= (\gamma' + \pi)i , & \mu(S_3, T_2; S_2) &= \gamma'' i ,
\end{aligned}$$

and the theorem takes the form:

Three lines, each through one of the vertices A, B, C of a triangle, have a common normal if and only if

$$\sin\alpha' \sin\beta' \sin\gamma' = -\sin\alpha'' \sin\beta'' \sin\gamma'' .$$

VI.7 The bisectors and radii of a right-angled hexagon

Let $(S_n, n \bmod 6)$ be a right-angled hexagon and $\mathbf{s}_n$ line-matrices determining the side-lines S_n. We assume that S_2, S_4, S_6 are proper and $\mathbf{s}_2, \mathbf{s}_4, \mathbf{s}_6$ normalized. Then the matrices $\mathbf{s}_2 + \mathbf{s}_6$, $\mathbf{s}_4 + \mathbf{s}_2$, $\mathbf{s}_6 + \mathbf{s}_4$ determine transversals which are the concordant bisectors of the double crosses $(S_6, S_2; S_1)$, $(S_2, S_4; S_3)$, $(S_4, S_6; S_5)$, respectively (cf. V.4). They will be called the *concordant bisectors* of the sides $\sigma_1, \sigma_3, \sigma_5$ and denoted by B_{c1}, B_{c3}, B_{c5}. The line matrices $\mathbf{s}_2 - \mathbf{s}_6$, $\mathbf{s}_4 - \mathbf{s}_2$, $\mathbf{s}_6 - \mathbf{s}_4$ determine the *reverse bisectors* of the sides $\sigma_1, \sigma_3, \sigma_5$. They will be denoted by B_{r1}, B_{r3}, B_{r5}. If the side-lines S_1, S_3, S_5 are proper, all this and what follows applies after interchange of the subscripts $2, 4, 6, ev$ with $5, 1, 3, od$, respectively.

The matrices $\mathbf{s}_2 - \mathbf{s}_6$, $\mathbf{s}_4 - \mathbf{s}_2$, $\mathbf{s}_6 - \mathbf{s}_4$ being linearly dependent, but pairwise linearly independent, we have:

The reverse bisectors B_{r1}, B_{r3}, B_{r5} of a right-angled hexagon $(S_n, n \bmod 6)$ with S_2, S_4, S_6 proper have a unique common normal B_{od} called the *bisector axis of the odd-numbered sides.*

If B_{r1}, B_{r3}, B_{r5} are proper, this follows also from the fact that the product $b_5 \circ b_3 \circ b_1$ of the half-turns about these lines maps S_6 orientation-reversing onto itself and, consequently, is a half-turn (cf. IV.2).

We note that one reverse bisector and two-condordant bisectors also have a common normal. This is, however, covered by the statement above applied to the same hexagon with the orientation of one of the lines S_2, S_4, S_6 reversed.

We claim that

$$(1) \qquad \mathbf{b}_{od} = \frac{\mathbf{s}_1}{\operatorname{tr}(\mathbf{s}_1\mathbf{s}_4)} + \frac{\mathbf{s}_3}{\operatorname{tr}(\mathbf{s}_3\mathbf{s}_6)} + \frac{\mathbf{s}_5}{\operatorname{tr}(\mathbf{s}_5\mathbf{s}_2)}$$

is a line matrix determining B_{od} up to orientation. (As mentioned at the beginning of VI.6, the denominators do not vanish.) Indeed, it is easily verified that

$$\operatorname{tr}(\mathbf{b}_{od}(\mathbf{s}_2 - \mathbf{s}_6)] = 0\,, \quad \operatorname{tr}[\mathbf{b}_{od}(\mathbf{s}_4 - \mathbf{s}_2)] = 0\,, \quad \operatorname{tr}[\mathbf{b}_{od}(\mathbf{s}_6 - \mathbf{s}_4)] = 0\,.$$

If all side-lines are proper, we have (cf. VI.2) $\operatorname{tr}(\mathbf{s}_1\mathbf{s}_4) = -2\cosh\alpha_{14}$, $\operatorname{tr}(\mathbf{s}_3\mathbf{s}_6) = -2\cosh\alpha_{36}$, $\operatorname{tr}(\mathbf{s}_5\mathbf{s}_2) = -2\cosh\alpha_{52}$, where $\alpha_{14}, \alpha_{36}, \alpha_{52}$ denote the altitudes, determined up to sign, of the hexagon. Hence, the matrix

$$(2) \qquad -2\,\mathbf{b}_{od} = \frac{\mathbf{s}_1}{\cosh\alpha_{14}} + \frac{\mathbf{s}_3}{\cosh\alpha_{36}} + \frac{\mathbf{s}_5}{\cosh\alpha_{52}}$$

also determines B_{od}. Using VI.5(2) and its analogues, we see that this also holds for the matrices

$$2\,i\,\mathbf{b}_{od}\,am(S_2, S_4, S_6) = \mathbf{s}_1 \sinh\sigma_1 + \mathbf{s}_3 \sinh\sigma_3 + \mathbf{s}_5 \sinh\sigma_5$$

and

$$2\,i\,\mathbf{b}_{od}\,am(S_1, S_3, S_5) = \mathbf{s}_1 \sinh\sigma_4 + \mathbf{s}_3 \sinh\sigma_6 + \mathbf{s}_5 \sinh\sigma_2\,.$$

Clearly, $\mathbf{b}_{od}$ may also be written as a linear combination of $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_6$. The coefficients are, however, rather involved.

We shall show that, in a sense to be made precise, B_{od} has equal complex distances from the lines S_2, S_4, S_6.

Assume first that B_{r1}, B_{r3}, B_{r5} are proper, and let b_{r1}, b_{r3}, b_{r5} as before, denote the half-turns about these lines. With an arbitrarily chosen orientation of B_{od}, if proper, we have

$$\begin{aligned} b_{r1}(S_6) &= -S_2\,, & b_{r3}(S_2) &= -S_4\,, & b_{r5}(S_4) &= -S_6\,,\\ b_{r1}(B_{od}) &= -B_{od} & b_{r3}(B_{od}) &= -B_{od}\,, & b_{r5}(B_{od}) &= -B_{od}\,. \end{aligned}$$

Since the lines S_2, S_4, S_6 are distinct, none of them can coincide with B_{od}. Consequently there are unique common normals N_2 of S_2 and B_{od}, N_4 of S_4 and B_{od}, N_6 of S_6 and B_{od}. Any two of N_2, N_4, N_6 are mapped onto each other by one of the half-turns, and they are therefore all proper or all improper. (Actually, the last case is easily seen to be impossible under the assumption.) Choose an orientation of N_6, if proper, and orient N_2 and N_4 such that

$$b_{r1}(N_6) = N_2\,, \quad b_{r3}(N_2) = N_4\,, \quad b_{r5}(N_4) = N_6\,,$$

the last equation following from the fact that $b_{r5} \circ b_{r3} \circ b_{r1}$ is the half-turn about N_6. Since the widths of the double crosses $(S_n, B_{od}; N_n)$ satisfy

$$\mu(-S_n, -B_{od}; N_n) = \mu(S_n, B_{od}; N_n) \quad \text{for} \quad n = 2, 4, 6$$

the properties listed of the half-turns imply that they are equal each other. Their common value

$$\varrho_{ev} = \mu(S_2, B_{od}; N_2) = \mu(S_4, B_{od}; N_4) = \mu(S_6, B_{od}; N_6) \tag{3}$$

will be called the *radius of the even-numbered sides*. Its geometrical significance is this: If B_{od} is proper, S_2, S_4, S_6 are tangents of the equidistant cylinder with axis B_{od} and radius $|\operatorname{Re}\varrho_{ev}|$, and they form equal angles, $\operatorname{Im}\varrho_{ev}$ (depending on the orientation of B_{od}), with the generators (oriented in accordance with B_{od}) of the cylinder. If B_{od} is improper, we have $\varrho_{ev} = +\infty$ or $-\infty$. There is a horosphere with centre at B_{od} and passing through the intersection of S_2 and N_2. It touches S_2 and, as application of the half-turns b_3 and b_2 shows, also S_4 and S_6. Further, in the sense of the Euclidean geometry on the horosphere, they have the same direction at the points of contact.

Suppose now that improper lines occur among B_{r1}, B_{r3}, B_{r5}. We shall show that ϱ_{ev} also in this case can be defined by (2). If, for instance, B_{r1} is improper, then either the initial points or the terminal points of S_6 and S_2 coincide with $B_{r1} = S_1$. If both B_{r3} and B_{r5} were also improper, any two of S_2, S_4, S_6 should have a common end and be concordantly oriented at it. Since these ends, S_1, S_3, S_5, are distinct, this is impossible. Therefore we may assume that B_{r3}, say, is proper. Further, B_{od} must be proper. Otherwise B_{r3} would have an end at

$B_{od} = B_{r1} = S_1$ and consequently coincide with the common normal S_2 of S_1 and S_3, which is impossible since S_2 and S_4 do not coincide. Observing that N_2, N_4, N_6 are here improper, we see that

$$\mu(S_6, B_{od}; N_6) = \mu(S_2, B_{od}; N_2) = 0 \quad \text{or} \quad \pi i$$

depending on the orientation of B_{od}. Applying the half-turn b_3 to $(S_2, B_{od}; N_2)$, we see that $\mu(S_4, B_{od}; N_4)$ has the same value. We infer: If a reverse bisector is improper, the side-lines S_2, S_4, S_6 are parallel to the bisector axis B_{od} (one of them coinciding with B_{od} if two reverse bisectors are improper). The radius ϱ_{ev} equals 0 or πi.

We are now going to derive an expression for ϱ_{ev} in terms of the odd-numbered sides, valid whenever S_2, S_4, S_6 and B_{od} are proper. Actually we shall consider $\tanh^2 \varrho_{ev}$ which is independent of the irrelevant orientations of B_{od}, B_2, N_4, N_6.

We apply the relation at the end of V.4 to the lines $L_0 = B_{od}$, $L_1 = S_2$, $L_2 = S_4$, $L_3 = S_6$. This gives

$$\begin{vmatrix} 1 & \cosh \varrho_{ev} & \cosh \varrho_{ev} & \cosh \varrho_{ev} \\ \cosh \varrho_{ev} & 1 & \cosh \sigma_1 & \cosh \sigma_3 \\ \cosh \varrho_{ev} & \cosh \sigma_1 & 1 & \cosh \sigma_5 \\ \cosh \varrho_{ev} & \cosh \sigma_3 & \cosh \sigma_5 & 1 \end{vmatrix} = 0\,.$$

The left-hand side may be written

$$\begin{vmatrix} \cosh^2 \varrho_{ev} & \cosh \varrho_{ev} & \cosh \varrho_{ev} & \cosh \varrho_{ev} \\ \cosh \varrho_{ev} & 1 & \cosh \sigma_1 & \cosh \sigma_3 \\ \cosh \varrho_{ev} & \cosh \sigma_1 & 1 & \cosh \sigma_5 \\ \cosh \varrho_{ev} & \cosh \sigma_3 & \cosh \sigma_5 & 1 \end{vmatrix}$$

$$- \begin{vmatrix} \cosh^2 \varrho_{ev} - 1 & \cosh \varrho_{ev} & \cosh \varrho_{ev} & \cosh \varrho_{ev} \\ 0 & 1 & \cosh \sigma_1 & \cosh \sigma_3 \\ 0 & \cosh \sigma_1 & 1 & \cosh \sigma_5 \\ 0 & \cosh \sigma_3 & \cosh \sigma_5 & 1 \end{vmatrix}$$

$$= \cosh^2 \varrho_{ev} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \cosh \sigma_1 & \cosh \sigma_3 \\ 1 & \cosh \sigma_1 & 1 & \cosh \sigma_5 \\ 1 & \cosh \sigma_3 & \cosh \sigma_5 & 1 \end{vmatrix} - (\cosh^2 \varrho_{ev} - 1)\, am^2(S_2, S_4, S_6)$$

$$= 2\cosh^2 \varrho_{ev} (\cosh \sigma_1 - 1)(\cosh \sigma_3 - 1)(\cosh \sigma_5 - 1) - \sinh^2 \varrho_{ev}\, am^2(S_2, S_4, S_6)\,.$$

Hence we obtain

$$(4) \qquad \tanh^2 \varrho_{ev} = \frac{2(\cosh\sigma_1 - 1)(\cosh\sigma_3 - 1)(\cosh\sigma_5 - 1)}{am^2(S_2, S_4, S_6)}.$$

Using $\cosh\xi - 1 = 2\sinh^2 \xi/2$ and I.2(4), this may be written

$$(5) \quad \tanh^2 \varrho_{eb} = \frac{[\sinh\sigma_{od} - \sinh(\sigma_{od} - \sigma_1) - \sinh(\sigma_{od} - \sigma_3) - \sinh(\sigma_{od} - \sigma_5)]^2}{4\sinh\sigma_{od}\sinh(\sigma_{od} - \sigma_1)\sinh(\sigma_{od} - \sigma_3)\sinh(\sigma_{od} - \sigma_5)}.$$

If all side-lines of the hexagon are proper, $\tanh^2 \varrho_{ev}$ can be expressed in terms of the even-numbered sides by means of VI.5(8):

$$(6) \qquad \begin{aligned} \tanh^2 \varrho_{ev} &= \frac{4\sinh^2 \varrho_{ev}}{am^2(S_1, S_3, S_5)} \\ &= \frac{\sinh \varrho_{ev}}{\sinh(\sigma_{ev} - \sigma_2)\sinh(\sigma_{ev} - \sigma_4)\sinh(\sigma_{ev} - \sigma_6)}. \end{aligned}$$

Using VI.5(9), we get

$$(7) \qquad \tanh^2 \varrho_{ev} = -\frac{\tanh^2 \frac{\sigma_1}{2} \tanh^2 \frac{\sigma_3}{2} \tanh^2 \frac{\sigma_5}{2}}{\sinh^2 \sigma_{ev}},$$

and applying VI.2(9) to the last expression in (6),

$$(8) \qquad \begin{aligned} \tanh^2 \sigma_{ev} &= -\frac{\tanh^2 \frac{\sigma_1}{2}}{\sinh^2(\sigma_{ev} - \sigma_4)} = -\frac{\tanh^2 \frac{\sigma_3}{2}}{\sinh^2(\sigma_{ev} - \sigma_6)} \\ &= -\frac{\tanh^2 \frac{\sigma_5}{2}}{\sinh^2(\sigma_{ev} - \sigma_2)}. \end{aligned}$$

These relations admit of a simple geometrical interpretation and proof. Consider again the reverse bisectors B_{r1}, B_{r3}, B_{r5}, the bisector axis B_{od}, these lines oriented arbitrarily, and the common normals N_2, N_4, N_6 of B_{od} and S_2, S_4, S_6, respectively, these lines oriented concordantly, as described above.

Observing that

$$\mu(S_6, B_{r1}; S_1) + \mu(B_{r1}, S_2; S_1) = \sigma_1$$

and that application of the half-turn b_{r1} about B_{r1} yields

$$\mu(B_{r1}, S_2; S_1) = \mu(B_{r1}, -S_6; -S_1) = \mu(S_6, B_{r1}; S_1) - \pi i,$$

we find

$$\mu(S_6, B_{r1}; S_1) = \frac{\sigma_1}{2} + \frac{\pi}{2} i, \quad \mu(B_{r1}, S_2; S_1) = \frac{\sigma_1}{2} - \frac{\pi}{2} i$$

with one of the two values of $\frac{\sigma_1}{2}$ (with the other one if the orientation of B_{r1} is reversed). Analogously we get

$$\mu(S_2, B_{r3}; S_3) = \frac{\sigma_3}{2} + \frac{\pi}{2} i, \quad \mu(B_{r3}, S_4; S_3) = \frac{\sigma_3}{2} - \frac{\pi}{2} i,$$

$$\mu(S_4, B_{r5}; S_5) = \frac{\sigma_5}{2} + \frac{\pi}{2} i, \quad \mu(B_{r5}, S_6; S_5) = \frac{\sigma_5}{2} - \frac{\pi}{2} i.$$

With the notations

$$\begin{aligned}
\sigma_2' &= \mu(S_1, N_2; S_2), \quad & \sigma_4' &= \mu(S_3, N_4; S_4), \quad & \sigma_6' &= \mu(S_5, N_6; S_6), \\
\sigma_2'' &= \mu(N_2, S_3; S_2), \quad & \sigma_4'' &= \mu(\mathrm{N}_4, S_5; S_4), \quad & \sigma_6'' &= \mu(N_6, S_1; S_6),
\end{aligned}$$

we have

$$\sigma_2' + \sigma_2'' = \sigma_2, \quad \sigma_4' + \sigma_4'' = \sigma_4, \quad \sigma_6' + \sigma_6'' = \sigma_6.$$

Application of the half-turn b_{r3} about B_{r3} shows that

$$\begin{aligned}
\sigma_2'' &= \mu(N_2, S_3; S_2) = \mu(N_4, -S_3; -S_4) \\
&= \mu(S_3, N_4; S_4) + \pi i = \sigma_4' + \pi i.
\end{aligned}$$

Analogously one gets

$$\sigma_4'' = \sigma_6' + \pi i, \qquad \sigma_6'' = \sigma_2' + \pi i.$$

Hence,

$$\sigma_2' + \sigma_4' = \sigma_2 - \pi i, \quad \sigma_4' + \sigma_6' = \sigma_4 - \pi i, \quad \sigma_6' + \sigma_2' = \sigma_6 - \pi i,$$

and thus

$$\sigma_2' + \sigma_4' + \sigma_6' = \sigma_{ev} - \frac{3}{2} \pi i$$

with one of the values of

$$\sigma_{ev} = \frac{1}{2}(\sigma_2 + \sigma_4 + \sigma_6)$$

(with the other one if the orientations of N_2, N_4, N_6 are reversed). These relations yield

$$\sigma_2' = \sigma_{ev} - \sigma_4 - \frac{\pi}{2} i, \quad \sigma_4' = \sigma_{ev} - \sigma_6 - \frac{\pi}{2} i, \quad \sigma_6' = \sigma_{ev} - \sigma_2 - \frac{\pi}{2} i,$$

$$\sigma_2'' = \sigma_{ev} - \sigma_6 + \frac{\pi}{2}i\,, \quad \sigma_4'' = \sigma_{ev} - \sigma_2 + \frac{\pi}{2}i\,, \quad \sigma_6'' = \sigma_{ev} - \sigma_4 + \frac{\pi}{2}i\,.$$

Consider now the right-angled pentagon $(B_{r1}, S_1, S_2, N_2, B_{od})$. Three consecutive sides of it are

$$\mu(B_{r1}, S_2; S_1) = \frac{\sigma_1}{2} - \frac{\pi}{2}i\,, \quad \mu(S_1, N_2; S_2) = \sigma_2'\,,$$
$$\mu(S_2, B_{od}; N_2) = \varrho_{ev}\,,$$

and VI.2(5) yields

$$\cosh\sigma_2' = -\coth\left(\frac{\sigma_1}{2} - \frac{\pi}{2}i\right)\coth\varrho_{ev}\,.$$

From this and the analogues for σ_4' and σ_6' we obtain

$$\tanh\varrho_{ev} = -\frac{i\tanh\dfrac{\sigma_1}{2}}{\sinh(\sigma_{ev} - \sigma_4)} = -\frac{i\tanh\dfrac{\sigma_3}{2}}{\sinh(\sigma_{ev} - \sigma_6)}$$
$$= -\frac{i\tanh\dfrac{\sigma_5}{2}}{\sinh(\sigma_{ev} - \sigma_2)}\,.$$

Here the orientations of B_{r1}, B_{r3}, B_{r5} and B_{od} do not matter. If those of N_2, N_4, N_6 are reversed, ϱ_{ev} changes sign, in accordance with the change of σ_{ev} by πi.

We specialize the preceding results to triangles. As in VI.3(5), we consider a triangle with vertices A, B, C as a right-angled hexagon (S_n, n mod 6) the odd-numbered side-lines of which are the side-lines of the triangle, and the even-numbered are normals to the plane of the triangle through the vertices. In contrast to VI.3(5) it is of interest here to allow for several choices of the orientations of the S_n. In addition to the one in VI.3.5) we shall consider two others.

The plane P of the triangles is assumed to be oriented.

I. The side-lines S_1, S_3, S_5 of the triangle are concordantly oriented in the positive sense, and S_2, S_4, S_6 are the positive normals of P through C, A, P respectively. With the usual notations we then have

$$\sigma_1 = a\,, \qquad \sigma_3 = b\,, \qquad \sigma_5 = c\,,$$
$$\sigma_4 = (\pi - \alpha)i\,, \qquad \sigma_6 = (\pi - \beta)i\,, \qquad \sigma_2 = (\pi - \gamma)i\,.$$

Defining the positive numbers s and σ by

$$2s = a + b + c\,, \qquad 2\sigma = \alpha + \beta + \gamma\,,$$

we obtain

$$\sigma_{od} = s\,,$$

$$\sigma_{od} - \sigma_1 = s - a\,, \qquad \sigma_{od} - \sigma_3 = s - b\,, \qquad \sigma_{od} - \sigma_5 = s - c\,,$$

$$\sigma_{ev} = \left(\frac{3}{2}\pi - \sigma\right)i\,,$$

$$\sigma_{ev} - \sigma_2 = \left(\frac{\pi}{2} - \sigma + \gamma\right)i\,, \qquad \sigma_{ev} - \sigma_4 = \left(\frac{\pi}{2} - \sigma + \alpha\right)i\,,$$

$$\sigma_{ev} - \sigma_6 = \left(\frac{\pi}{2} - \sigma + \beta\right)i\,.$$

II. As under I, but with the orientation of one of the even-numbered sides, S_2 say, reversed. Then

$$\sigma_1 = a + \pi i\,, \qquad \sigma_3 = b + \pi i\,, \qquad \sigma_5 = c\,,$$
$$\sigma_4 = (\pi - \alpha)i\,, \quad \sigma_6 = (\pi - \beta)i\,, \quad \sigma_2 = (\gamma - \pi)i\,,$$

$$\sigma_{od} = s + \pi i\,,$$

$$\sigma_{od} - \sigma_1 = s - a\,, \qquad \sigma_{od} - \sigma_3 = s - b\,, \qquad \sigma_{od} - \sigma_5 = s - c + \pi i\,,$$

$$\sigma_{ev} = \left(\frac{\pi}{2} - \sigma + \gamma\right)i\,,$$

$$\sigma_{ev} - \sigma_2 = \left(\frac{3\pi}{2} - \sigma\right)i\,, \qquad \sigma_{ev} - \sigma_4 = \left(-\frac{\pi}{2} + \sigma - \beta\right)i\,,$$

$$\sigma_{ev} - \sigma_6 = \left(-\frac{\pi}{2} + \sigma - \alpha\right)i\,.$$

III. As under I, but with the orientation of one of the odd-numbered sides, S_5 say, reversed. Then

$$\sigma_1 = a\,, \qquad \sigma_3 = b\,, \qquad \sigma_5 = -c\,,$$
$$\sigma_4 = -\alpha i\,, \ \sigma_6 = -\beta i\,, \ \sigma_2 = (\pi - \gamma)i\,,$$

$$\sigma_{od} = s - c\,,$$

$$\sigma_{od} - \sigma_1 = b - s\,, \qquad \sigma_{od} - \sigma_3 = a - s\,, \qquad \sigma_{od} - \sigma_5 = s\,,$$

$$\sigma_{ev} = \left(\frac{\pi}{2} - \sigma\right)i\,,$$

$$\sigma_{ev} - \sigma_2 = (\gamma - \sigma)i\,, \qquad \sigma_{ev} - \sigma_4 = \left(\frac{\pi}{2} + \alpha - \sigma\right)i\,,$$

$$\sigma_{ev} - \sigma_6 = \left(\frac{\pi}{2} + \beta - \sigma\right)i\,.$$

Assuming the vertices to be proper, we consider first the reverse bisectors B_{r1}, B_{r3}, B_{r5} in case I. They are the bisecting normals, lying in the plane P, of the sides BC, CA, AB, respectively. Since they have a common normal B_{od}, they are either concurrent and B_{od} is orthogonal to P, or they have B_{od} as a common end, or they are ultraparallel and B_{od} lies in P. In the first case the triangle has a *circumscribed circle*. If $R > 0$ denotes the radius of the latter, we have $\varrho_{ev} = \pm R$. In the second case the triangle has a *circumscribed horocycle* and $\varrho_{ev} = \pm\infty$. In the third case the triangle has a *circumscribed equidistant curve*. Then $\varrho_{ev} = \pm D \pm \frac{\pi}{2} i$, where $D > 0$ denotes the distance of the latter from its axis B_{od}. Hence, the three cases occur according as $\tanh^2 \varrho_{ev} < 1$, $\tanh^2 \varrho_{ev} = 1$, or $\tanh^2 \varrho_{ev} > 1$.

The expression (4) for $\tanh^2 \varrho_{ev}$ may here be written

$$\tanh^2 \varrho_{ev} = \frac{16 \sinh^2 \frac{a}{2} \sinh^2 \frac{b}{2} \sinh^2 \frac{c}{2}}{am_v^2}.$$

Because of VI.5(7), the denominator equals

$$16 \sin^2 \frac{a}{2} \sinh^2 \frac{b}{2} \sinh^2 \frac{c}{2} +$$
$$+ 4 \left(\sin \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2} \right) \left(- \sinh \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2} \right)$$
$$\cdot \left(\sinh \frac{a}{2} - \sinh \frac{b}{2} + \sinh \frac{c}{2} \right) \left(\sinh \frac{a}{2} + \sinh \frac{b}{2} - \sinh \frac{c}{2} \right),$$

and hence:

A triangle with sides a, b, c has a circumscribed circle, a circumscribed horocycle, or a circumscribed equidistant curve according as each of the quantities $\sinh \frac{a}{2}$, $\sinh \frac{b}{2}$, $\sinh \frac{c}{2}$ *is less than the sum of the others, one of them equals the sum of the others, or one of them is greater than the sum of the others.*

Specializing (4) and (6) we obtain the following expressions for R or D in terms of the sides and the angles:

$$\text{(9)} \quad \left.\begin{matrix} \tanh^2 R \\ 1 \\ \coth^2 D \end{matrix}\right\} = \frac{2(\cosh a - 1)(\cosh b - 1)(\cosh c - 1)}{1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2 \cosh a \cosh b \cosh c}$$
$$= \frac{\cos \sigma}{\cos(\sigma - \alpha) \cos(\sigma - \beta) \cos(\sigma - \gamma)},$$

and (8) may here be written

$$(10)\qquad \left.\begin{matrix}\tanh R\\ 1\\ \coth D\end{matrix}\right\} = \frac{\tanh\dfrac{a}{2}}{\cos(\sigma-\alpha)} = \frac{\tanh\dfrac{b}{2}}{\cos(\alpha-\beta)} = \frac{\tanh\dfrac{c}{2}}{\cos(\sigma-\gamma)}.$$

We consider briefly the specialization II in which the orientation of S_2, that is the line orthogonal to P through C, is reversed. The bisectors B_{r1} and B_{r3} are now the normals to P through the midpoints of the sides, BC and CA, respectively, while B_{r5} as before, is the bisecting normal in P of AB. Hence, the line joining the midpoints of BC and CA is the bisector axis B_{od} and intersects the bisecting normal of AB at right angles. The vertices have the same distance D_C from this line. Since $\varrho_{ev} = \pm D_C \pm \dfrac{\pi}{2} i$, we obtain from (4) and (6)

$$(11)\qquad \begin{aligned}\coth^2 D_C &= \frac{2(\cosh a+1)(\cosh b+1)(\cosh c-1)}{1-\cosh^2 a-\cosh^2 b-\cosh^2 c+2\cosh a\cosh b\cosh c}\\ &= \frac{\cos(\sigma-\gamma)}{\cos\sigma\cos(\sigma-\alpha)\cos(\sigma-\beta)},\end{aligned}$$

and (8) yields

$$(12)\qquad \coth D_C = \frac{\coth\dfrac{a}{2}}{\cos(\alpha-\beta)} = \frac{\coth\dfrac{b}{2}}{\cos(\sigma-\alpha)} = \frac{\tanh\dfrac{c}{2}}{\cos\sigma}$$

There are simple direct proofs for the statement above and the last relations.

We return to the specialization I, admitting now improper vertices of the triangle. The reverse bisectors B_{r2}, B_{r4}, B_{r6} are the bisectors of the interior angles at C, A, B, respectively. They are concurrent, their common point being the centre of the *inscribed circle* of the triangle. If $r > 0$ denotes its radius, we have $\varrho_{od} = \pm r \pm \dfrac{\pi}{2} i$, hence $\tanh^2 \varrho_{od} = \coth^2 r$. Therefore (4), with even and odd subscripts interchanged, and VI.5(13) yield

$$(13)\qquad \begin{aligned}\tanh^2 r &= -\frac{am_s^2}{2(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)}\\ &= \frac{2\cos\sigma\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)}{(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)}.\end{aligned}$$

Since $-\sigma < \sigma-\alpha \le \sigma$, $-\sigma \le \sigma-\beta \le \sigma$, $-\sigma \le \sigma-\gamma \le \sigma$, the numerator is greater than or equal to $2\cos^4\sigma$. Consequently,

(14) $\tanh r \geq \frac{1}{2}\cos^2\sigma$.

Using (5), we can also write

$$\tanh^2 r = \frac{4\cos\sigma\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)}{[\cos\sigma+\cos(\sigma-\alpha)+\cos(\sigma-\beta)+\cos(\alpha-\gamma)]^2}.$$

Applying the inequality between the arithmetic and geometric means to the denominator, we obtain

$$\tanh^2 r \leq \frac{1}{4}[\cos\sigma\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)]^{\frac{1}{2}}$$

and thus

(15) $\tanh r \leq \frac{1}{2}$

with equality if and only if $\alpha = \beta = \gamma = 0$, that is for the triangle with all vertices improper.

If the vertices are proper, (6), with even and odd subscripts interchanged, yields an expression for r in terms of the sides:

(16) $$\tanh^2 r = \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)}{\sinh s}.$$

Relations (8) take here the form

(17) $$\tanh r = \sinh(s-a)\tan\frac{\alpha}{2} = \sinh(s-b)\tan\frac{\beta}{2} = \sinh(s-c)\tan\frac{\gamma}{2}.$$

Finally we consider the specialization III, where the orientation of S_5, that is, the side AB of the triangle is reversed. We admit again improper vertices. The reverse bisector B_{r2} is here also the bisector of the interior angle at C, while B_{r4} and B_{r6} are now the bisectors of the exterior angles at A and B, respectively, improper if the vertex in question is improper. If these bisectors are concurrent, their common point is the centre of an *escribed circle* touching the side AB and prolongations of the other sides. Denoting the radius by $r_c > 0$, we have $\varrho_{od} = \pm r_c \pm \frac{\pi}{2}i$, hence $\tanh^2\varrho_{od} = \coth^2 r_c > 1$. If the bisectors have a common end, this is the centre of an *escribed horocycle*. In this case $\varrho_{od} = \pm\infty$ and thus $\tanh^2\varrho_{od} = 1$. If the bisectors are ultraparallel, their common normal is the axis of an *escribed equidistant curve*. Denoting its distance by $d_c \geq 0$, we have $\varrho_{od} = \pm d_c$ and thus $\tanh^2\varrho_{od} = \tanh^2 d_c < 1$. We recall that $d_c = 0$, if A or B or both are improper. Then we have a line "touching" the side lines at ends.

Interchanging odd and even subscripts in (3) we get here with $\sigma_2 = (\pi - \gamma) i$, $\sigma_4 = -\alpha i$, $\sigma_6 = -\beta i$

$$\left.\begin{matrix} \coth^2 r_c \\ 1 \\ \tanh^2 d_c \end{matrix}\right\} = \frac{-2(1-\cos\alpha)(1-\cos\beta)(1+\cos\gamma)}{am_s^2} \tag{18}$$

$$= \frac{-16\sin^2\dfrac{\alpha}{2}\sin^2\dfrac{\beta}{2}\cos^2\dfrac{\gamma}{2}}{am_s^2},$$

according as the right-hand side is greater than, equal to, or less than 1. Now, with the values of σ_2, σ_4, σ_6 above, VI.5(7) takes the form

$$\begin{aligned} am_s^2 = &-16\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\cos^2\frac{\gamma}{2} + \\ &+4\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}-\cos\frac{\gamma}{2}\right)\left(-\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\cos\frac{\gamma}{2}\right) \\ &\cdot\left(\sin\frac{\alpha}{2}-\sin\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\cos\frac{\gamma}{2}\right). \end{aligned}$$

Consequently, the sign of the last term determines which of the three cases takes place. The last factor in the term is clearly positive, and so are the second and the third. To see this, observe that $\gamma < \pi - \alpha - \beta$ and therefore $\cos\frac{\gamma}{2} > \sin\frac{\alpha+\beta}{2}$. Hence

$$\begin{aligned} -\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\cos\frac{\gamma}{2} &> -\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\alpha+\beta}{2} \\ &= 2\left(\sin\frac{\beta-\alpha}{4}+\sin\frac{\alpha+\beta}{4}\right)\cos\frac{\alpha+\beta}{4} \geq 0, \end{aligned}$$

and analogously for the third factor. This yields:

A triangle ABC has an escribed circle, horocycle, or equidistant curve beyond the side AB according as

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2} \gtreqless \cos\frac{\gamma}{2}.$$

Applying VI.5(13) to the expression (18), one obtains $\coth^2 r_c$ or $\tanh^2 d_c$ in terms of the angles. If all three vertices are proper, (6) with odd and even sub-

scripts interchanged yields

$$(19)\qquad \left.\begin{matrix}\coth^2 r_c\\ 1\\ \tanh^2 d_c\end{matrix}\right\} = \frac{\sinh(s-c)}{\sin s \sinh(s-a)\sinh(s-b)},$$

and (8) may here be written

$$(20)\qquad \left.\begin{matrix}\tanh^2 r_c\\ 1\\ \coth^2 d_c\end{matrix}\right\} = \sinh s \tan\frac{\gamma}{2} = \sinh(s-a)\cot\frac{\beta}{2} = \sinh(s-b)\cot\frac{\alpha}{2}.$$

Let $(S_n, n \bmod 6)$ be a right-angled hexagon with proper side-lines and $\mathbf{s}_n, n \bmod 6$, normalized line matrices determining them. We consider now the concordant bisectors B_{c1}, B_{c3}, B_{c5}. They have no common normal since the line matrices $\mathbf{s}_6+\mathbf{s}_2, \mathbf{s}_2+\mathbf{s}_4, \mathbf{s}_4+\mathbf{s}_6$ determining them are linearly independent. Denote by b_{c1}, b_{c3}, b_{c5} the half-turns about B_{c1}, B_{c3}, B_{c5}, respectively, and by

$$\beta_{od} = \delta(b_{c5}\circ b_{c3}\circ b_{c1})$$

any one of the two opposite values of the displacement of the motion $b_{c5}\circ b_{c3}\circ b_{c1}$. It is easily seen that β_{od} is independent of the order in which the half-turns are taken. We shall prove that

$$(21)\qquad \pm\beta_{od} = \sigma_2+\sigma_4+\sigma_6+\pi i\,.$$

The motion $g = b_{c5}\circ b_{c3}\circ b_{c1}$ is determined by the matrix

$$\mathbf{g} = (\mathbf{s}_4+\mathbf{s}_6)(\mathbf{s}_2+\mathbf{s}_4)(\mathbf{s}_6+\mathbf{s}_2)\,.$$

Using $\mathbf{s}_n^2 = -\mathbf{1}$ and $\operatorname{tr}\mathbf{s}_n = 0$, one easily finds

$$\operatorname{tr}\mathbf{g} = 2\operatorname{tr}(\mathbf{s}_6\mathbf{s}_4\mathbf{s}_2) = -4\,am(S_2,S_4,S_6)$$

and

$$\begin{aligned}\det\mathbf{g} &= \det(\mathbf{s}_6+\mathbf{s}_4)\det(\mathbf{s}_2+\mathbf{s}_4)\det(\mathbf{s}_6+\mathbf{s}_2)\\ &= (2-\operatorname{tr}(\mathbf{s}_6\mathbf{s}_4))(2-\operatorname{tr}(\mathbf{s}_2\mathbf{s}_4))(2-\operatorname{tr}(\mathbf{s}_6\mathbf{s}_2))\\ &= 8(1+\cosh\sigma_5)(1+\cosh\sigma_3)(1+\cosh\sigma_1).\end{aligned}$$

Now IV.2(1) applied to $\mathbf{f}^2 = \mathbf{g}^2/\det\mathbf{g}$ yields

$$\begin{aligned}\cosh\beta_{od} &= \frac{\operatorname{tr}^2\mathbf{g}}{2\det\mathbf{g}} - 1\\ &= \frac{am^2(S_2,S_4,S_6)}{(1+\cosh\sigma_5)(1+\cosh\sigma_3)(1+\cosh\sigma_1)} - 1\,.\end{aligned}$$

Because of VI.5(10), the right-hand side equals

$$-2\sinh^2\sigma_{ev}-1=-\cosh(\sigma_2+\sigma_4+\sigma_6),$$

and this proves the statement (21).

There is also a simple geometric argument leading to (21). The definition of the concordant bisectors shows that $g=b_{c5}\circ b_{c3}\circ b_{c1}$ maps S_6 orientation-preserving onto itself, so S_6 is the axis of g. We have to determine the width of the double cross $(S_1, g(S_1); S_6)$. Since $b_{c1}(S_1)=-S_1$, the double cross $(b_{c1}(S_1), S_3; S_2)$ has width $\sigma_2+\pi i$. Its image by b_{c3} is $(b_{c3}\circ b_{c1}(S_1), -S_3; S_4)$. Hence, the double cross $(b_{c3}\circ b_{c1}(S_1), S_5; S_4)$ has width $\sigma_2+\sigma_4$ since the width of $(-S_3, S_5; S_4)$ is $\sigma_4-\pi i$. Application of b_{c5} gives $(g(S_1), -S_5; S_6)$. Since $(-S_5, S_1; S_6)$ has width $\sigma_6+\pi i$, the width of $(g(S_1), S_1; S_6)$ is $\sigma_2+\sigma_4+\sigma_6+\pi i$. Choosing the value of δ_{od} in accordance with the orientation of S_6, wo obtain

$$\delta_{od}=-\sigma_2-\sigma_4-\sigma_6-\pi i$$

and thus the statement.

The specialization I to a triangle ABC with proper vertices and restriction to its plane yields the following:

The product of the half-turns about the midpoints of the sides BC, CA, AB, applied in this order, is a rotation with centre B through the angle $\alpha+\beta+\gamma$.

Specialization II yields:

The product of the reflections in the orthogonal bisectors of the sides BC and CA and the half-turn about the midpoint of AB, applied in this order, is a rotation with centre B through the angle $\alpha+\beta-\gamma$.

Returning to specialization I but considering now the bisectors B_{c4}, B_{c6}, B_{c2} we obtain:

The product of the reflections in the bisectors of the exterior angles at A, B, C, applied in this order, is a glide-reflection under which its axis CA is translated the distance $a+b+c$.

Specialization III gives here:

The product of the reflections in the bisectors of the interior angles at A and B and the reflection in the bisector of the exterior angle at C is a glide-reflection under which its axis CA is translated the distance $a+b-c$.

All these statements about triangles may easily be verified directly.

VI.8 The medians of a right-angled hexagon

We consider again a right-angled hexagon $(S_n, n \bmod 6)$ with proper even-numbered sides. Let $\mathbf{s}_n$ as before denote the normalized line matrix determining S_n if S_n is proper, otherwise any line matrix determining S_n. The co-transversals

(cf. VI.6) of the concordant bisectors B_{c1}, B_{c3}, B_{c5}, that is, the common normals of B_{c1} and S_4, of B_{c3} and S_6, and of B_{c5} and S_2, may be considered as generalizations of the medians of a triangle. They will be denoted by M_1, M_3, M_5, respectively, and be called the *medians of the odd-numbered sides.* If S_1, S_3, S_5 are proper, the medians M_2, M_4, M_6 of the even-numbered sides are defined analogously.

According to VI.6(2), line matrices determining M_1, M_3, M_5 are

$$\begin{aligned} \mathbf{m}_1 &= -\operatorname{tr}(\mathbf{s}_5\mathbf{s}_2)\mathbf{s}_3 + \operatorname{tr}(\mathbf{s}_3\mathbf{s}_6)\mathbf{s}_5\,, \\ \mathbf{m}_3 &= -\operatorname{tr}(\mathbf{s}_1\mathbf{s}_4)\mathbf{s}_5 + \operatorname{tr}(\mathbf{s}_5\mathbf{s}_2)\mathbf{s}_1\,, \\ \mathbf{m}_5 &= -\operatorname{tr}(\mathbf{s}_3\mathbf{s}_6)\mathbf{s}_1 + \operatorname{tr}(\mathbf{s}_1\mathbf{s}_4)\mathbf{s}_3\,. \end{aligned} \tag{1}$$

From their obvious linear dependence or the theorem in VI.6 one infers that M_1, M_3, M_5 have a unique common normal M_{od}, the *median axis* of the odd-numbered sides.

A line matrix determining M_{od} is

$$\mathbf{m}_{od} = \mathbf{s}_2 + \mathbf{s}_4 + \mathbf{s}_6\,. \tag{2}$$

Indeed, $\mathbf{s}_2 + \mathbf{s}_4, \mathbf{s}_6$ and $\mathbf{m}_{od}$ are linearly dependent. Hence, the line determined by $\mathbf{m}_{od}$ intersects the common normal M_3 of B_{c3} and S_6 orthogonally. Analogously one argues for M_1 and M_5.

Observing that

$$\det(\mathbf{s}_2 + \mathbf{s}_6) = 2 - \operatorname{tr}(\mathbf{s}_2\mathbf{s}_6) = 2(1 + \cosh\sigma_1) = 4\cosh^2\frac{\sigma_1}{2}\,,$$

we see that $(\mathbf{s}_2 + \mathbf{s}_6)\Big/\left(2\cosh\frac{\sigma_1}{2}\right)$ for any choice of one of the two values of $\frac{\sigma_1}{2}$ is the normalized line matrix determining B_{c1} with a certain orientation. Writing

$$\mathbf{m}_{od} = \mathbf{s}_4 + 2\cosh\frac{\sigma_1}{2}\,\frac{\mathbf{s}_2 + \mathbf{s}_6}{2\cosh\frac{\sigma_1}{2}}\,, \tag{3}$$

we conclude:

The median axis M_{od} divides the double crosses $(S_4, B_{c1}; M_1)$, $(S_6, B_{c3}; M_3)$, $(S_2, B_{c5}; M_5)$ in the ratios $-2\cosh\frac{\sigma_1}{2}$, $-2\cosh\frac{\sigma_3}{2}$, $-2\cosh\frac{\sigma_5}{2}$, respectively.

If μ_n denotes the width of the double cross $(S_{n+3}, B_{cn}; M_n)$ that is, the "complex length" of the median M_n, we have

$$\cosh^2\mu_1 = \frac{\operatorname{tr}^2[\mathbf{s}_4(\mathbf{s}_2 + \mathbf{s}_6)]}{4\det(\mathbf{s}_2 + \mathbf{s}_6)} = \frac{(\cosh\sigma_3 + \cosh\sigma_5)^2}{2(\cosh\sigma_1 + 1)} \tag{4}$$

and the analogues for μ_3 und μ_5.

Let μ'_n, n odd, denote the width of the double cross $(S_{n+3}, M_{od}; M_n)$ that is the complex distance of the median axis M_{od} from the side-line S_{n+3}. Then (cf. (2))

$$\begin{aligned}\cosh^2 \mu'_1 &= \frac{\operatorname{tr}^2 [\mathbf{s}_4(\mathbf{s}_2 + \mathbf{s}_4 + \mathbf{s}_6)]}{4 \det(\mathbf{s}_2 + \mathbf{s}_4 + \mathbf{s}_6)} \\ &= \frac{[\operatorname{tr}(\mathbf{s}_4\mathbf{s}_2) - 2 + \operatorname{tr}(\mathbf{s}_4\mathbf{s}_6)]^2}{4[3 - \operatorname{tr}(\mathbf{s}_2\mathbf{s}_4) - \operatorname{tr}(\mathbf{s}_4\mathbf{s}_6) - \operatorname{tr}(\mathbf{s}_6\mathbf{s}_2)]} \qquad (5) \\ &= \frac{(1 + \cosh\sigma_3 + \cosh\sigma_5)^2}{3 + 2\cosh\sigma_1 + 2\cosh\sigma_3 + 2\cosh\sigma_5},\end{aligned}$$

and the analogues for μ'_3 and μ'_5.

We specialize again to triangles. In the specialization I described in the preceding section the concordant bisectors B_{c1}, B_{c3}, B_{c5} are the normals to the plane P of the triangle ABC through the midpoints A', B', C' of the sides BC, CA, AB, respectively. Hence, M_1, M_3, M_5 are the lines AA', BB', CC', and thus, the medians in the usual sense. They are concurrent. Their common point G is usually called the *centroid* of the triangle.

Since $\sigma_1 = a$, $\sigma_3 = b$, $\sigma_5 = c$, we obtain from (4) for the positive length m_a of the median AA'

$$\cosh m_a = \frac{\cosh b + \cosh c}{2\cosh \dfrac{a}{2}}.$$

According to (5) the distance m'_a between A and G is determined by

$$\cosh^2 m'_a = \frac{(1 + \cosh b + \cosh c)^2}{3 + 2\cosh a + 2\cosh b + 2\cosh c}.$$

From (3) we infer that G divides AA' (with any orientation) in the ratio

$$\frac{\sinh AG}{\sinh A'G} = -2\cosh\frac{a}{2}.$$

Consider now the specialization II with the orientation of S_2 reversed, so $\sigma_1 = a + \pi i$, $\sigma_3 = b + \pi i$, $\sigma_5 = c$. The concordant bisectors B_{c1} and B_{c3} are now the bisecting normals in the plane P of the sides BC and CA, while B_{c5}, as before, is normal to P. The median M_1 is therefore the line in P through A and orthogonal to B_{c1}, the median M_3 the line through B orthogonal to B_{c3}, and M_5 as before the line joining C and the midpoint C' of AB. These three lines have a common normal M_{od}. They may be concurrent, or have a common end, or be ultraparallel with a common normal in P. In the first case the common point G_c is called an *ex-centroid* of the triangle.

Let μ'_1, μ'_3, μ'_5 denote the widths of the double crosses $(S_4, M_{od}; M_1)$,

$(S_6, M_{od}; M_3)$, $(S_2, M_{od}; M_5)$. In the first case they are real or have the imaginary part π, hence $\cosh^2 \mu_n' > 1$ for $n = 1, 3, 5$; in the second case $\cosh^2 \mu_n' = \infty$; and in the third the μ_n' have imaginary part $\pm \frac{\pi}{2}$, hence $\cosh^2 \mu_n' < 0$. Now (5) and the relations obtained by permuting the subscripts 1, 3, 5 cyclically yield

$$\cosh^2 \mu_1' = \frac{(1 - \cosh b + \cosh c)^2}{3 - 2\cosh a - 2\cosh b + 2\cosh c},$$

$$\cosh^2 \mu_3' = \frac{(1 - \cosh a + \cosh c)^2}{3 - 2\cosh a - 2\cosh b + 2\cosh c},$$

$$\cosh^2 \mu_5' = \frac{(1 - \cosh a - \cosh b)^2}{3 - 2\cosh a - 2\cosh b + 2\cosh c}.$$

Consequently, a necessary and sufficient condition for the existence of the excentroid G_c is

$$2(\cosh a + \cosh b) < 3 + 2\cosh c .$$

(It is easily checked that the fractions on the right-hand sides acutally are greater than 1 if the condition is satisfied. For the third one the triangle inequality $a + b > c$ has to be used.)

We consider briefly the medians of the even-numbered sides. In specialization I the concordant bisectors B_{c2}, B_{c4}, B_{c6} are the bisectors of the exterior angles at the vertices C, A, B, respectively. The medians M_2, M_4, M_6 are the common normals of these bisectors with the opposite side-lines AB, BC, CA, respectively. Since the medians have a common normal, we infer in particular:

If each of the bisectors of the exterior angles of a triangle intersects the opposite side-line, the three intersection points are collinear.

In specialization III where the orientation of S_5, that is, the side-line AB, is reversed, the concordant bisectors B_{c4} and B_{c6} are the bisectors of the interior angles at A and B, respectively, while B_{c2} is the bisector of the exterior angle at C. Hence we have:

If the bisector of the exterior angle at a vertex of a triangle intersects the opposite side-line, the intersection point is collinear with the points at which the bisectors of the interior angles at the other vertices intersect the opposite side-lines.

To find a condition for that the bisector of the exterior angle at the vertex C intersects the line AB we may consider any of the specializations I or III. Using I, the bisector in question is B_{c2} and $AB = S_5$. If these lines intersect, M_2 is the normal to the plane P of the triangle through the point of intersection. Consequently, M_{ev} is a line in P through this point. This amounts to $\mu_2' = (S_5, M_{ev}; M_2)$ being purely imaginary and, thus, to $0 < \cosh^2 \mu_2' < 1$. Now (5) with the odd and even subscripts interchanged yields, because of $\sigma_2 = (\pi - \gamma)i$, $\sigma_4 = (\pi - \alpha)i$,

$\sigma_6 = (\pi - \beta) i$,

$$\cosh^2 \mu_2' = \frac{(1 - \cos\alpha - \cos\beta)^2}{3 - 2\cos\alpha - 2\cos\beta - 2\cos\gamma}.$$

The condition in question is therefore

$$(1 - \cos\alpha - \cos\beta)^2 < 3 - 2\cos\alpha - 2\cos\beta - 2\cos\gamma,$$

which may be reduced to

$$\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} < \sin\frac{\gamma}{2}.$$

VI.9 The altitudes of a right-angled hexagon

The altitude lines A_{14}, A_{36}, A_{52} of a right-angled hexagon (S_n, $n \bmod 6$) are by definition common normals of opposite side-lines S_1 and S_4, S_3 and S_6, S_5 and S_2, respectively (cf. VI.2). They are uniquely determined unless two opposite side-lines coincide. In this section we exclude the latter case. To see what this means for matrices $\mathbf{s}_n$ determining the side-lines S_n, we observe the following. If, for instance, S_1 and S_4 coincide, we have because of $S_n \perp S_{n+1}$, $n \bmod 6$ (where $\perp$ means that the two lines intersect orthogonally) that $S_3 \perp S_1$, $S_5 \perp S_1$, $S_2 \perp S_4$, $S_6 \perp S_4$. Conversely, any two of these relations imply that S_1 and S_4 coincide. Assume, for instance, $S_3 \perp S_1$ and $S_2 \perp S_4$. Since also $S_2 \perp S_1$ and $S_3 \perp S_4$, it follows that S_1 as well as S_4 is the common normal of S_2 and S_3. Similarly the other cases are dealt with. This shows that the requirement that no two opposite side-lines coincide is satisfied if and only if $S_n \perp S_{n+2}$ for at most one value of $n \bmod 6$, hence if and only if

$$\text{(1)} \qquad \operatorname{tr}(\mathbf{s}_n \mathbf{s}_{n+2}) = 0 \quad \text{for at most one} \quad n \bmod 6.$$

A line matrix determining A_{14} may be written $p\mathbf{s}_3 + q\mathbf{s}_5, p, q \in \mathbb{C}$, since A_{14} is a transversal of the double cross $(S_3, S_5; S_4)$. Because of $A_{14} \perp S_1$ we must have

$$\operatorname{tr}(\mathbf{s}_1 (p\mathbf{s}_3 + q\mathbf{s}_5)] = p \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_3) + q \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_5) = 0$$

and we may choose

$$p = \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_5), \qquad q = -\operatorname{tr}(\mathbf{s}_1 \mathbf{s}_3).$$

Hence,

$$\text{(2)} \qquad \mathbf{a}_{14} = \mathbf{s}_3 \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_5) - \mathbf{s}_5 \operatorname{tr}(\mathbf{s}_3 \mathbf{s}_1),$$

which does not vanish because of (1), is a line matrix determining A_{14}. Analogously one gets line matrices determining A_{36} and A_{52}:

$$\mathbf{a}_{36} = \mathbf{s}_5 \operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) - \mathbf{s}_1 \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3),$$
$$\mathbf{a}_{52} = \mathbf{s}_1 \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3) - \mathbf{s}_3 \operatorname{tr}(\mathbf{s}_1\mathbf{s}_5).$$

The matrices obtained by replacing the subscripts 1, 3, 5 by 4, 6, 2, respectively, determine also the lines A_{14}, A_{36}, A_{52}.

Obviously, the matrices $\mathbf{a}_{14}$, $\mathbf{a}_{36}$, $\mathbf{a}_{52}$, but no two of them, are linearly dependent. We have therefore the Hjelmslev-Morley Theorem:

The altitude lines of a right-angled hexagon with no two opposite side-lines coinciding have a unique common normal.

This common normal O will be called the *orthoaxis* of the hexagon. A line matrix determining it is

$$\begin{aligned}
\mathbf{o} &= \mathbf{a}_{36}\mathbf{a}_{14} - \mathbf{a}_{14}\mathbf{a}_{36} \\
&= [\mathbf{s}_5 \operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) - \mathbf{s}_1 \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3)]\,[\mathbf{s}_3 \operatorname{tr}(\mathbf{s}_1\mathbf{s}_5) - \mathbf{s}_5 \operatorname{tr}(\mathbf{s}_3\mathbf{s}_1)] \\
&\quad - [\mathbf{s}_3 \operatorname{tr}(\mathbf{s}_1\mathbf{s}_5) - \mathbf{s}_5 \operatorname{tr}(\mathbf{s}_3\mathbf{s}_1)]\,[\mathbf{s}_5 \operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) - \mathbf{s}_1 \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3)] \\
&= (\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5) \operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) \operatorname{tr}(\mathbf{s}_1\mathbf{s}_5) \qquad (3) \\
&\quad + (\mathbf{s}_1\mathbf{s}_5 - \mathbf{s}_5\mathbf{s}_1) \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3) \operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) \\
&\quad + (\mathbf{s}_3\mathbf{s}_1 - \mathbf{s}_1\mathbf{s}_3) \operatorname{tr}(\mathbf{s}_1\mathbf{s}_5) \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3).
\end{aligned}$$

Since $\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5$ is a line matrix for the common normal S_4 of S_3 and S_5, we must have

$$\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5 = c\mathbf{s}_4$$

for some $c \in \mathbb{C}$. Multiplying by $\mathbf{s}_1$ from the right, taking traces, and observing that $\operatorname{tr}(\mathbf{s}_4\mathbf{s}_1) \neq 0$, we obtain

$$c = \frac{2 \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)}{\operatorname{tr}(\mathbf{s}_4\mathbf{s}_1)}.$$

This, the analogues for $\mathbf{s}_1\mathbf{s}_5 - \mathbf{s}_5\mathbf{s}_1$ and $\mathbf{s}_3\mathbf{s}_1 - \mathbf{s}_1\mathbf{s}_3$, and $\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1) \neq 0$ yield

$$\begin{aligned}
\frac{1}{2\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)}\mathbf{o} &= \mathbf{s}_4 \frac{\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1)\operatorname{tr}(\mathbf{s}_1\mathbf{s}_5)}{\operatorname{tr}(\mathbf{s}_4\mathbf{s}_1)} \\
&\quad + \mathbf{s}_6 \frac{\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3)\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1)}{\operatorname{tr}(\mathbf{s}_6\mathbf{s}_3)} \qquad (4) \\
&\quad + \mathbf{s}_2 \frac{\operatorname{tr}(\mathbf{s}_1\mathbf{s}_5)\,(\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3)}{\operatorname{tr}(\mathbf{s}_2\mathbf{s}_5)}.
\end{aligned}$$

Here improper side-lines may occur. If all side-lines are proper, VI.5(2) and the

analogues may be used to obtain

$$\begin{aligned}\frac{1}{8}\mathbf{o} = {} & \mathbf{s}_4 \sinh\sigma_4 \cosh\sigma_2 \cosh\sigma_6 \\ & + \mathbf{s}_6 \sinh\sigma_6 \cosh\sigma_4 \cosh\sigma_2 \\ & + \mathbf{s}_2 \sinh\sigma_2 \cosh\sigma_6 \cosh\sigma_4 \,.\end{aligned}$$

If in addition $\cosh\sigma_2 \cosh\sigma_4 \cosh\sigma_6 \neq 0$, the matrix

$$\mathbf{s}_2 \tanh\sigma_2 + \mathbf{s}_4 \tanh\sigma_4 + \mathbf{s}_6 \tanh\sigma_6$$

also determines the orthoaxis O.

Interchange of the subscripts 1, 3, 5 with 2, 4, 6 in the expressions for $\mathbf{o}$ yields another matrix determining O.

We admit again improper side-lines, but assume that S_1 is a proper one. We are going to determine the complex distance ω_1 of the lines O and S_1. As will be shown below, $(O, S_1; A_{14})$, with arbitrarily chosen orientations of O and A_{14} if proper, is a double cross as defined in V.3. Here this means that if O is improper, it does not coincide with an end of S_1, equivalently, if $\det\mathbf{o} = 0$, then $\operatorname{tr}(\mathbf{s}_1\mathbf{o}) \neq 0$. Defining ω_1 to be the width of this double cross, we have

$$\cosh^2\omega_1 = \frac{\operatorname{tr}^2(\mathbf{s}_1\mathbf{o})}{4\det\mathbf{o}}$$

also if O is improper in which case this gives the correct value $\cosh^2\omega_1 = \infty$. To prove the claim we suppose that O is improper and coincides with an end of S_1, that is $\operatorname{tr}(\mathbf{s}_1\mathbf{o}) = 0$. Observing that $\operatorname{tr}(\mathbf{s}_1\mathbf{s}_6) = \operatorname{tr}(\mathbf{s}_1\mathbf{s}_2) = 0$, we obtain from (4)

$$\text{(5)} \qquad \operatorname{tr}(\mathbf{s}_1\mathbf{o}) = 2\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1)\operatorname{tr}(\mathbf{s}_1\mathbf{s}_5)\,.$$

Since $\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1) \neq 0$, the assumption implies that one of the two last factors, $\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1)$ say, vanishes and so $S_1 \perp S_3$. Since O is an end of S_1, their common normal A_{14} is improper and coincides with O. This implies that S_4 has at least one end at O, and S_3, being the common normal of S_1 and S_4, must be improper and coincide with the end O of S_1. This is, however, excluded by the definition of a hexagon (cf. VI.1).

If S_1, S_3, S_5 are proper, $\cosh^2\omega_1$ can be expressed in terms of the even-numbered sides. For $\operatorname{tr}(\mathbf{s}_1\mathbf{o})$ this is evident since (5) may be written

$$\operatorname{tr}(\mathbf{s}_1\mathbf{o}) = -16\,am(S_1, S_3, S_5)\cosh\sigma_2\cosh\sigma_6\,.$$

To compute $\det\mathbf{o}$ it is convenient to use the expression (3) for $\mathbf{o}$. Writing out only one of three analogous terms, we have

$$\begin{aligned}\det\mathbf{o} = {} & \det(\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5)\operatorname{tr}^2(\mathbf{s}_3\mathbf{s}_1)\operatorname{tr}^2(\mathbf{s}_1\mathbf{s}_5) + \ldots + \ldots \\ & + \operatorname{tr}[(\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5)(\mathbf{s}_1\mathbf{s}_5 - \mathbf{s}_5\mathbf{s}_1)^{\sim}]\operatorname{tr}^2(\mathbf{s}_3\mathbf{s}_5)\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3) \\ & + \ldots + \ldots\end{aligned}$$

Now

$$\det(\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5) = 2 - \operatorname{tr}(\mathbf{s}_3\mathbf{s}_5)^2 = 4 - \operatorname{tr}^2(\mathbf{s}_5\mathbf{s}_3)$$
$$= 4(1 - \cosh^2\sigma_4)$$

and the analogues, further

$$\begin{aligned}\operatorname{tr}[\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5)(\mathbf{s}_1\mathbf{s}_5 - \mathbf{s}_5\mathbf{s}_1)^{\sim}] &= \operatorname{tr}[(\mathbf{s}_5\mathbf{s}_3 - \mathbf{s}_3\mathbf{s}_5)(\mathbf{s}_5\mathbf{s}_1 - \mathbf{s}_1\mathbf{s}_5)] \\ &= \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_5\mathbf{s}_1 + \mathbf{s}_3\mathbf{s}_5\mathbf{s}_1\mathbf{s}_5) + 2\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) \\ &= 2\operatorname{tr}(\mathbf{s}_5\mathbf{s}_3)\operatorname{tr}(\mathbf{s}_5\mathbf{s}_1) + 4\operatorname{tr}(\mathbf{s}_3\mathbf{s}_1) \\ &= 8(\cosh\sigma_4\cosh\sigma_6 - \cosh\sigma_2)\end{aligned}$$

and the analogues. Hence

$$\begin{aligned}\frac{1}{64}\det\mathbf{o} = {} & (1 - \cosh^2\sigma_4)\cosh^2\sigma_2\cosh^2\sigma_6 \\ & + (1 - \cosh^2\sigma_6)\cosh^2\sigma_4\cosh^2\sigma_2 \\ & + (1 - \cosh^2\sigma_2)\cosh^2\sigma_6\cosh^2\sigma_4 \\ & + 2(\cosh\sigma_4\cosh\sigma_6 - \cosh\sigma_2)\cosh^2\sigma_2\cosh\sigma_6\cosh\sigma_4 \\ & + 2(\cosh\sigma_6\cosh\sigma_2 - \cosh\sigma_4)\cosh^2\sigma_4\cosh\sigma_2\cosh\sigma_6 \\ & + 2(\cosh\sigma_2\cosh\sigma_4 - \cosh\sigma_6)\cosh^2\sigma_6\cosh\sigma_4\cosh\sigma_2\end{aligned}$$

which reduces to

$$\begin{aligned}\frac{1}{64}\det\mathbf{o} = {} & \cosh^2\sigma_4\cosh^2\sigma_6 + \cosh^2\sigma_6\cosh^2\sigma_2 + \cosh^2\sigma_2\cosh^2\sigma_4 \\ & - 2\cosh\sigma_2\cosh\sigma_4\cosh\sigma_6(\cosh^2\sigma_2 + \cosh^2\sigma_4 + \cosh^2\sigma_6) \\ & + 3\cosh^2\sigma_2\cosh^2\sigma_4\cosh^2\sigma_6\,.\end{aligned}$$

Using VI.5(6), one verifies that this can be written

$$\begin{aligned}\frac{1}{64}\det\mathbf{o} = {} & (2\cosh\sigma_2\cosh\sigma_4\cosh\sigma_6 - 1)\,am^2(S_1, S_3, S_5) \\ & - \sinh^2\sigma_2\sinh^2\sigma_4\sinh^2\sigma_6\,.\end{aligned}$$

We therefore obtain

$$\cosh^2\omega_1 =$$
$$= \frac{am^2(S_1, S_3, S_5)\cosh^2\sigma_2\cosh^2\sigma_6}{(2\cosh\sigma_2\cosh\sigma_4\cosh\sigma_6 - 1)\,am^2(S_1, S_3, S_5) - \sinh^2\sigma_2\sinh^2\sigma_4\sinh^2\sigma_6}\,.$$

This will be used to derive a necessary and sufficient condition for the altitude lines of a triangle to be concurrent. Under the specialization frequently used we have

$$\sigma_2 = (\pi - \gamma)i, \quad \sigma_4 = (\pi - \alpha)i, \quad \sigma_6 = (\pi - \beta)i$$

and $am(S_1, S_3, S_5) = am_s$. Hence we get here

$$\cosh^2 \omega_1 = \frac{am_s^2 \cos^2 \beta \cos^2 \gamma}{\sin^2 \alpha \sin^2 \beta \sin^2 \gamma - (2 \cos \alpha \cos \beta \cos \gamma + 1)\, am_s^2} .$$

The altitude lines are concurrent, that is, O orthogonal to the plane of the triangle, if and only if $\operatorname{Im} \omega_1 = \pm \frac{1}{2}\pi$, thus $\cosh^2 \omega_1 \leq 0$. Since $am_s^2 < 0$ (cf. VI.5(11)), the condition in question is

$$(2 \cos \alpha \cos \beta \cos \gamma + 1)\, am_s^2 < \sin^2 \alpha \sin^2 \beta \sin^2 \gamma ,$$

covering also the case $\cos \beta \cos \gamma = 0$. It is obviously satisfied if $2 \cos \alpha \cos \beta \cos \gamma + 1 \geq 0$, in particular if the greatest angle is not greater than $\frac{2}{3}\pi$.

The co-transversals of the altitude lines are also of interest. Clearly, an altitude line cannot coincide with one of the side-lines of which it is the common normal, unless both are improper. This does not occur either. Assume, for instance, $A_{14} = S_1$ improper. Then S_4 would have an end at this improper point, and consequently S_3 would be improper and coincide with S_1. This is, however, excluded by the definition of a right-angled hexagon. For each $n \bmod 6$ there is therefore a unique line A_n^* normal to S_n and $A_{n,n+3}$. It will be called the *co-altitude line* of the side-line S_n.

In the following we consider the co-altitude lines of the odd-numbered side-lines. A line matrix determining A_1^* can be written

$$\mathbf{a}_1^* = p\mathbf{s}_6 + q\mathbf{s}_2 , \qquad p, q \in \mathbb{C} .$$

It has to satisfy $\operatorname{tr}(\mathbf{a}_1^* \mathbf{a}_{14}) = 0$, hence because of (2),

$$\begin{aligned} &\operatorname{tr}[p\mathbf{s}_6 + q\mathbf{s}_2)(\mathbf{s}_3 \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_5) - \mathbf{s}_5 \operatorname{tr}(\mathbf{s}_3 \mathbf{s}_1))] \\ &= p \operatorname{tr}(\mathbf{s}_6 \mathbf{s}_3) \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_5) - q \operatorname{tr}(\mathbf{s}_2 \mathbf{s}_5) \operatorname{tr}(\mathbf{s}_3 \mathbf{s}_1) = 0 . \end{aligned}$$

We may therefore choose

$$\text{(6)} \qquad \mathbf{a}_1^* = \mathbf{s}_6 \operatorname{tr}(\mathbf{s}_2 \mathbf{s}_5) \operatorname{tr}(\mathbf{s}_3 \mathbf{s}_1) + \mathbf{s}_2 \operatorname{tr}(\mathbf{s}_6 \mathbf{s}_3) \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_5)$$

and for A_3^* and A_5^*

$$\begin{aligned} \mathbf{a}_3^* &= \mathbf{s}_2 \operatorname{tr}(\mathbf{s}_4 \mathbf{s}_1) \operatorname{tr}(\mathbf{s}_5 \mathbf{s}_3) + \mathbf{s}_4 \operatorname{tr}(\mathbf{s}_2 \mathbf{s}_5) \operatorname{tr}(\mathbf{s}_3 \mathbf{s}_1) , \\ \mathbf{a}_5^* &= \mathbf{s}_4 \operatorname{tr}(\mathbf{s}_6 \mathbf{s}_3) \operatorname{tr}(\mathbf{s}_1 \mathbf{s}_5) + \mathbf{s}_6 \operatorname{tr}(\mathbf{s}_4 \mathbf{s}_1) \operatorname{tr}(\mathbf{s}_5 \mathbf{s}_3) . \end{aligned}$$

Whether these three matrices are linearly independent depends on the

determinant

$$\begin{vmatrix} \mathrm{tr}(\mathbf{s}_6\mathbf{s}_3)\,\mathrm{tr}(\mathbf{s}_1\mathbf{s}_5) & 0 & \mathrm{tr}(\mathbf{s}_2\mathbf{s}_5)\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1) \\ \mathrm{tr}(\mathbf{s}_4\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_3) & \mathrm{tr}(\mathbf{s}_2\mathbf{s}_5)\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1) & 0 \\ 0 & \mathrm{tr}(\mathbf{s}_6\mathbf{s}_3)\,\mathrm{tr}(\mathbf{s}_1\mathbf{s}_5) & \mathrm{tr}(\mathbf{s}_4\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_3) \end{vmatrix}$$

$$= 2\,\mathrm{tr}(\mathbf{s}_6\mathbf{s}_3)\,\mathrm{tr}(\mathbf{s}_2\mathbf{s}_5)\,\mathrm{tr}(\mathbf{s}_4\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_1\mathbf{s}_5)\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_3)\,.$$

Because of VI.6(1) it can only vanish if one (cf.(1)) and only one of the three last factors does. In this case two of A_1^*, A_3^*, A_5^* coincide and are different from the third. Assume, for instance, $\mathrm{tr}(\mathbf{s}_5\mathbf{s}_1) = 0$. Then $(S_1, S_2, S_3, S_4, S_5)$ is a right-angled pentagon, A_{14} coincides with S_5, A_{52} with S_1, and $A_1^* = A_5^* = S_6$. Not all what follows is valid in this "special case". This will be pointed out. In the "general case", where the determinant is different from zero, A_1^*, A_3^*, A_5^* have no common normal and an improper among them can not coincide with an end of one of the others. To see this, suppose A_1^* is improper and coincides with an end of A_3^*. Then either S_1 or A_{14} must be improper and coincide with A_1^*. If S_1 is improper, S_2 must be proper and end at $S_1 = A_1^*$. The common normal S_3 of S_2 and A_3^* must then also coincide with S_1, which is excluded by the definition of the hexagon. If A_{14} is improper, S_1 and S_4 must end at $A_{14} = A_1^*$ and therefore S_3, the common normal of S_4 and A_3^*, must be improper and coincide with A_1^* and thus with an end of S_1. This is, however, also excluded. Hence, in the general case the lines A_1^*, A_2^*, A_5^* satisfy the conditions of the odd-numbered side-lines of a right-angled hexagon.

We assume now the side-lines S_1, S_3, S_5 to be proper and denote by s_1, s_3, s_5 the half-turns about these lines. We claim that the axis A_{35}^* of the motion $s_3 \circ s_1 \circ s_5$ is the common normal of A_3^* and A_5^* (or a normal in the special case where $A_3^* = A_5^*$), the axis A_{51}^* of $s_5 \circ s_3 \circ s_1$ is a common normal of A_5^* and A_1^*, and the axis A_{13}^* of $s_1 \circ s_5 \circ s_3$ a common normal of A_1^* and A_3^*. These statements, that is,

$$\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5\mathbf{a}_3^*) = \mathrm{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5\,[\mathbf{s}_2\,\mathrm{tr}(\mathbf{s}_4\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_3) + \mathbf{s}_4\,\mathrm{tr}(\mathbf{s}_2\mathbf{s}_5)\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1)]) = 0\,,$$

$$\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5\mathbf{a}_5^*) = \mathrm{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5\,[\mathbf{s}_4\,\mathrm{tr}(\mathbf{s}_6\mathbf{s}_3)\,\mathrm{tr}(\mathbf{s}_1\mathbf{s}_5) + \mathbf{s}_6\,\mathrm{tr}(\mathbf{s}_4\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_3)]) = 0$$

and the analogues, are immediate consequences of the following trace relations, where $\mathbf{s}_n\mathbf{s}_{n+1} = -\mathbf{s}_{n+1}\mathbf{s}_n$, $(\mathbf{s}_m\mathbf{s}_n)^{\sim} = \mathbf{s}_n\mathbf{s}_m$, and I.3(2) are used:

$$\mathrm{tr}(\mathbf{s}_2\mathbf{s}_5)\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1) = \mathrm{tr}(\mathbf{s}_2\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1) + \mathrm{tr}(\mathbf{s}_5\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1) = 2\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5\mathbf{s}_2)\,,$$

$$\mathrm{tr}(\mathbf{s}_4\mathbf{s}_1)\,\mathrm{tr}(\mathbf{s}_5\mathbf{s}_3) = \mathrm{tr}(\mathbf{s}_4\mathbf{s}_1\mathbf{s}_5\mathbf{s}_3) + \mathrm{tr}(\mathbf{s}_1\mathbf{s}_4\mathbf{s}_5\mathbf{s}_3) = -2\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5\mathbf{s}_4)\,,$$

$$\mathrm{tr}(\mathbf{s}_6\mathbf{s}_3)\,\mathrm{tr}(\mathbf{s}_1\mathbf{s}_5) = \mathrm{tr}(\mathbf{s}_6\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5) + \mathrm{tr}(\mathbf{s}_3\mathbf{s}_6\mathbf{s}_1\mathbf{s}_5) = 2\,\mathrm{tr}(\mathbf{s}_3\mathbf{s}_1\mathbf{s}_5\mathbf{s}_6)$$

and the analogues for $\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1$ and $\mathbf{s}_1\mathbf{s}_5\mathbf{s}_3$.

Since the half-turns s_1, s_3, s_5 are involutory, we have

$$\begin{aligned} s_5 \circ (s_3 \circ s_1 \circ s_5) \circ s_5 &= s_5 \circ s_3 \circ s_1\,, \\ s_1 \circ (s_5 \circ s_3 \circ s_1) \circ s_1 &= s_1 \circ s_5 \circ s_3\,, \\ s_3 \circ (s_1 \circ s_5 \circ s_3) \circ s_3 &= s_3 \circ s_1 \circ s_5\,. \end{aligned}$$

Hence, s_5 maps A^*_{35} onto A^*_{51}, s_1 maps A^*_{51} onto A^*_{13}, and s_3 maps A^*_{13} onto A^*_{35}. This implies that the lines A^*_{35}, A^*_{51}, A^*_{15} are either all proper or all improper. If they are proper, we choose an orientation of A^*_{35}, carry it over to A^*_{51} by s_5 so that $s_5(A^*_{35}) = A^*_{51}$, and orient A^*_{13} such that $s_1(A^*_{51}) = A^*_{13}$. Then we have also $s_3(A^*_{13}) = A^*_{35}$ with the orientation chosen. Indeed, A^*_{35} is the axis of the motion $s_3 \circ s_1 \circ s_5$; for the latter maps A^*_{35} onto itself and is no half-turn since S_1, S_3, S_5 have no common normal. Choosing orientations arbitrarily of the proper ones among the lines A^*_1, A^*_3, A^*_5, we have in the general case a right-angled hexagon $(A^*_1, A^*_{13}, A^*_3, A^*_{35}, A^*_5, A^*_{51})$. For the sake of brevity we call it the *A^*_{od}-hexagon* associated with $(S_n, n \bmod 6)$. Provided A^*_{13}, A^*_{35}, A^*_{51} are proper, the side-lines S_1, S_3, S_5 are the concordant bisectors of its sides A^*_1, A^*_3, A^*_5, and consequently the altitude lines A_{14}, A_{36}, A_{52} are the reverse bisectors.

In the special case the A^*_{od}-hexagon degenerates. If $\mathrm{tr}(\mathbf{s}_1\mathbf{s}_5) = 0$, that is, $S_1 \perp S_5$, and thus S_6 proper, we have, as mentioned,

$$A^*_1 = A^*_5 = S_6$$

with suitable orientations of A^*_1 and A^*_5. This implies that A_{36}, A^*_{13}, A^*_{35} coincide. With orientations as described above of A^*_{13}, A^*_{35}, A^*_{51}, in case these are proper, we therefore have

$$s_3(A^*_{13}) = -A^*_{13} = A^*_{35}$$

which implies

$$s_1(A^*_{13}) = s_5(A^*_{35}) = A^*_{51}\,.$$

Hence, if A^*_{13}, A^*_{35}, A^*_{51} are proper, S_1, S_3, S_5 may be considered as the condordant bisectors of the degenerate hexagon.

The hexagons $(S_n, n \bmod 6)$ considered so far have proper side-lines S_1, S_3, S_5 and not two coinciding opposite side-lines. We assume now in addition that no two opposite side-lines are parallel, equivalently, that the altitude lines A_{14}, A_{36}, A_{52} are proper. Since S_1, S_3, S_5 are proper, this implies that A^*_1, A^*_3, A^*_5 are proper. Hence, the sides

$$\alpha^*_{13} = \mu(A^*_1, A^*_3; A^*_{13})\,, \quad \alpha^*_{35} = \mu(A^*_3, A^*_5; A^*_{35})\,,$$
$$\alpha^*_{51} = \mu(A^*_5, A^*_1; A^*_{51})$$

of the A^*_{od}-hexagon, also if it is degenerate, are finite. We observe that the sum of these sides does not depend on the (arbitrary) orientations of A^*_1, A^*_3, A^*_5 because two of the sides increase by πi if the orientation of one of these lines is changed.

We are going to prove:

Let $(S_n, n \bmod 6)$ be a right-angled hexagon with proper side-lines S_1, S_3, S_5 and such that no two opposite side-lines coincide or are parallel. The product $s_5 \circ s_3 \circ s_1$ of the half-turns about the side-lines S_1, S_3, S_5 has the axis A^*_{51}, and if $\pm\delta_{od} = \delta(s_5 \circ s_3 \circ s_1)$ denotes its displacement,

$$\pm\delta_{od} = \alpha^*_{13} + \alpha^*_{35} + \alpha^*_{51} + \pi i\,, \tag{7}$$

$$am^2(S_1, S_3, S_5) = -\sinh^2 \frac{1}{2}(\alpha^*_{13} + \alpha^*_{35} + \alpha^*_{51})\,. \tag{8}$$

First we observe that (7) and (8) are equivalent. Indeed, IV.2(1) applied to $f = s_5 \circ s_3 \circ s_1$ yields

$$\cosh\delta_{od} = \frac{1}{2}\operatorname{tr}^2(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1) - 1 = 2\,am^2(S_1, S_3, S_5) - 1\,.$$

If $A^*_{13}, A^*_{35}, A^*_{51}$ and, thus all side-lines of the A^*_{od}-hexagon are proper and the latter is not degenerate, application of VI.7(21) to the A^*_{od}-hexagon instead of $(S_n, n \bmod 6)$ gives (7). In the special case, (7) is easily checked directly. Assume as above that $S_1 \perp S_5$. Then $s_1 \circ s_5 = s_6$, the half-turn about S_6. Hence, because of $\delta(s_3 \circ s_1 \circ s_5) = \delta(s_5 \circ s_3 \circ s_1)$,

$$\pm\delta_{od} = \delta(s_3 \circ s_6) = 2\mu(S_6, S_3; A_{36})\,.$$

Since in this case $A^*_1 = A^*_5 = S_6$ and $A^*_{13} = -A^*_{35} = A_{14}$, we have

$$\begin{aligned}\alpha^*_{51} &= \mu(A^*_5, A^*_1; A^*_{51}) = 0\,,\\ \alpha^*_{13} &= \mu(A^*_1, A^*_2; A^*_{13}) = \mu(A^*_3, A^*_5; A^*_{35}) = \alpha^*_{35}\,.\end{aligned}$$

Using that A^*_3 is the common normal of S_3 and A_{36}, we obtain

$$\mu(A^*_1, A^*_3; A^*_{13}) = \mu(S_6, A^*_3, A_{36}) = \mu(S_6, S_3; A_{36}) + \frac{1}{2}\pi i$$

if A^*_3 is suitably oriented, and hence (7).

Suppose now that $A^*_{13}, A^*_{35}, A^*_{51}$ are improper. Since A^*_{51} is the axis of $s_5 \circ s_3 \circ s_1$, a line matrix determining it is

$$\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1 - (\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)^\sim = \mathbf{s}_5\mathbf{s}_3\mathbf{s}_1 + \mathbf{s}_1\mathbf{s}_3\mathbf{s}_5\,.$$

It is improper if and only if

$$\begin{aligned}\det(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1 + \mathbf{s}_1\mathbf{s}_3\mathbf{s}_5) &= 2 - \operatorname{tr}(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)^2 = 4 - \operatorname{tr}^2(\mathbf{s}_5\mathbf{s}_3\mathbf{s}_1)\\ &= 4\,[(1 - am^2(S_1, S_3, S_5)] = 0\,,\end{aligned}$$

hence, if and only if $s_5 \circ s_3 \circ s_1$ is a parallel motion and so $\delta_{od} = 0$. The special case cannot occur here. If, for instance, $S_3 \perp S_5$, then $s_5 \circ s_3 \circ s_1 = s_4 \circ s_1$, and this to be a parallel motion would require that S_1 and S_4 are parallel, which is excluded by assumption. Consequently, the A^*_{od}-hexagon is here a triangle with improper vertices A^*_{13}, A^*_{35}, A^*_{51}. With coherent orientations of A^*_1, A^*_3, A^*_5 we therefore have $\alpha^*_{13} = \alpha^*_{35} = \alpha^*_{51} = \pi i$, and this gives (7) in the present case.

To motivate the exclusion of the case that two opposite lines of $(S_n, n \bmod 6)$ are parallel, we note that in this case both $+\infty$ and $-\infty$ occur among α^*_{13}, α^*_{35}, α^*_{51}; so the right-hand side of (7) gets meaningless.

We specialize again to triangles ABC with $BC = S_1$, $CA = S_3$, $AB = S_5$. Improper vertices are admitted. Let A', B', C' denote the feet of the altitudes on the sides BC, CA, AB, respectively. The A^*_{od}-hexagon is here the foot triangle $A'B'C'$. With suitable orientations of the side-lines of the latter the side-lines of ABC are the condordant bisectors, and the altitudes of ABC the reverse bisectors at the vertices A', B', C'. To be more precise, if the triangle ABC is acute-angled, the sides of $A'B'C'$ have to be oriented coherently, and the lines BC, CA, AB are the bisectors of the exterior angles at A', B', C', respectively. If ABC is right-angled, at C say, then the triangle $A'B'C'$ is degenerate, and the sides $B'C'$ and $C'A'$ coinciding with the altitude from C have to be oriented oppositely. If ABC is obtuse-angled, at C say, and $B'C'$ oriented from B' towards C', then $C'A'$ has to be oriented from C' towards A', but $A'B'$ from B' to A'. The lines BC and CA are the bisector of the interior angles at A' and B', and the line AB is the bisector of the exterior angle at C'. Let a', b', c' denote the positive lengths of the sides $B'C'$, $C'A'$, $A'B'$. Considering the restrictions of the motions occurring to the plane of the triangle, we may formulate the former results in the present case as follows:

The product of the reflections in the side-lines BC, CA, AB of an acute-angled triangle, taken in this order, is a glide-reflection the axis of which, the side-line $C'A'$ of the foot triangle, is translated through the perimeter $a' + b' + c'$ of the latter. If the triangle ABC has a right-angle at C, this holds with $a' = b'$, $c' = 0$. If the angle at C is obtuse, $a' + b' + c'$ has to be replaced by $a' + b' - c'$.

For an acute-angled triangle (7) may be written

$$am_s = i \sinh \tfrac{1}{2}(a' + b' + c') \, .$$

As mentioned as a consequence of VI.5(13), we have

$$-i\, am_s = \sinh \tfrac{1}{2}(a' + b' + c') \leq 2$$

and thus:

The perimeter of the foot triangle of an acute-angled triangle is less than or equal to $2 \log(2 + \sqrt{5})$.

Interchanging the roles of the even-numbered and the odd-numbered side-lines, we let now S_1, S_3, S_5 be the normals to the plane of a triangle ABC with proper vertices through A, B, C, repectively. The co-altitudes A^*_1, A^*_3, A^*_5 are then

lines in the plane of the triangle passing through A, B, C and orthogonal to the altitude lines A_{14}, A_{35}, A_{52}, respectively. Since the side-lines $A^*_{35}, A^*_{51}, A^*_{13}$ of the A^*_{od}-hexagon are pairwise images of each other under one of the half-turns s_1, s_3, s_5, either all three of them are proper normals to the plane of the triangle and oriented towards the same half-space, or all of them are improper, or all of them are lying in the plane. Hence, in the plane the A^*_{od}-hexagon yields either a triangle with all vertices proper, or a triangle with all vertices improper, or a right-angled hexagon. This hexagon must be convex. Otherwise one of the lines $A^*_{13}, A^*_{35}, A^*_{51}$ would separate the two others (cf. VI.1); but this is impossible since the side-line of the given triangle joining two of these lines is ultraparallel to the third. The orientations of $A^*_{13}, A^*_{35}, A^*_{51}$ have to be chosen in accordance with one of the senses in which the hexagon may be traversed. Let the A^*_{od}-hexagon in the first two cases be called the *circumscribed triangle* and in the third case the *circumscribed hexagon* of the triangle ABC.

If ABC has a circumscribed triangle, let $\alpha^*, \beta^*, \gamma^*$ denote the interior angles of the latter at $A^*_{35}, A^*_{51}, A^*_{13}$, respectively. Then with suitable orientations of A^*_1, A^*_3, A^*_5

$$\alpha^*_{35} = (\pi - \alpha^*)i\,, \quad \alpha^*_{51} = (\pi - \beta^*)i\,, \quad \alpha^*_{13} = (\pi - \gamma^*)i$$

and, since $am_v > 0$ (cf. VI.5(12)), (8) may be written

$$am_v = \cos\tfrac{1}{2}(\alpha^* + \beta^* + \gamma^*)\,.$$

If ABC has a circumscribed hexagon, let a^*, b^*, c^* denote the positive lengths of the sides on the side-lines $A^*_{35}, A^*_{51}, A^*_{13}$, respectively. Then, with suitable orientations of A^*_1, A^*_3, A^*_5

$$\alpha^*_{35} = a^* + \pi i\,, \quad \alpha^*_{51} = b^* + \pi i\,, \quad \alpha^*_{13} = c^* + \pi i$$

and (8) may be written

$$am_v = \cosh\tfrac{1}{2}(a^* + b^* + c^*)\,.$$

Altogether we can state:

A triangle ABC with proper vertices has a circumscribed triangle with proper vertices if

$$am_v < 1\,.$$

The product of the half-turns about A, B, C is a rotation through the sum of the angles of the circumscribed triangle.

If

$$am_v = 1\,,$$

ABC has a circumscribed triangle with improper vertices. The product of the half-turns about A, B, C is a parallel motion.

If

$$am_v > 1\,,$$

ABC has a circumscribed hexagon. The product of the half-turns about A, B, C is a translation through the sum of the lengths of those sides of the hexagon which do not contain A, B or C.

Finally we discuss the question to what extent a right-angled hexagon (S_n, $n \bmod 6$) is determined by its A^*_{od}-hexagon. We shall show:

Given any right-angled hexagon ($A^*_1, A^*_{13}, A^*_3, A^*_{35}, A^*_5, A^*_{51}$) with all side-lines proper, there exists one and up to the orientations of the side-lines, only one hexagon (S_n, $n \bmod 6$) for which it is the A^*_{od}-hexagon.

The side-lines S_1, S_3, S_5 must be the concordant bisectors of the sides A^*_1, A^*_3, A^*_5, respectively. Since these bisectors are proper and have no common normal, they determine a hexagon (S_n, $n \bmod 6$) up to the orientations of the side-lines. It satisfies the requirement, namely that the reverse bisectors of the sides A^*_1, A^*_3, A^*_5 are its altitude lines. This follows immediately from the fact that two concordant bisectors, say S_3 of A^*_3 and S_5 of A^*_5, and the reverse bisector of A^*_1, thus, the common normal of A^*_1 and S_1, have a common normal, and this is S_4.

The situation is essentially different if the given hexagon has improper side-lines $A^*_{13}, A^*_{35}, A^*_{51}$ and, thus, is a triangle with improper vertices:

Given a triangle with side-lines A^*_1, A^*_2, A^*_5 and improper vertices $A^*_{13}, A^*_{35}, A^*_{51}$, further a proper or improper line O with ends different from these vertices, there exists one and, up to orientations of the side-lines, only one hexagon (S_n, $n \bmod 6$) for which the given triangle is the A$^*_{od}$*-hexagon and the given line O the orthoaxis.*

Let A_{14}, A_{36}, A_{52} denote the common normals of O and A^*_1, A^*_3, A^*_5, respectively. The assumptions imply that they are proper. Line matrices determining them may be written

$$\begin{aligned} \mathbf{a}_{14} &= p_1 \mathbf{a}^*_{13} + q_1 \mathbf{a}^*_{51}\,, \\ \mathbf{a}_{36} &= p_3 \mathbf{a}^*_{35} + q_3 \mathbf{a}^*_{13}\,, \qquad p_n, q_n \in \mathbb{C} \setminus \{0\}\,, \\ \mathbf{a}_{52} &= p_5 \mathbf{a}^*_{51} + q_5 \mathbf{a}^*_{35}\,, \end{aligned}$$

where $\mathbf{a}^*_{13}$, $\mathbf{a}^*_{35}$, $\mathbf{a}^*_{51}$ are singular matrices determining A^*_{13}, A^*_{35}, A^*_{51}. Since A_{14}, A_{36}, A_{52} have the common normal O, we have (cf. VI.6)

$$(*) \qquad p_1 p_3 p_5 + q_1 q_3 q_5 = 0\,.$$

The lines S_1, S_3, S_5 have to be the common normals of A_{14} and A^*_1, of A_{36} and A^*_3, of A_{52} and A^*_5, respectively. We claim that they are determined by the

matrices

$$\mathbf{s}_1 = p_1 \mathbf{a}^*_{13} - q_1 \mathbf{a}^*_{51}$$
$$\mathbf{s}_3 = p_3 \mathbf{a}^*_{35} - q_3 \mathbf{a}^*_{13}$$
$$\mathbf{s}_5 = p_5 \mathbf{a}^*_{51} - q_5 \mathbf{a}^*_{35} \,.$$

Indeed, the line determined by $\mathbf{s}_n$ is clearly normal to A^*_n, $n = 1, 3, 5$, and since $\mathbf{a}^{*2}_{13} = \mathbf{a}^{*2}_{35} = \mathbf{a}^{*2}_{51} = 0$, we have

$$\operatorname{tr}\,[(p_1 \mathbf{a}^*_{13} + q_1 \mathbf{a}^*_{51})\,(p_1 \mathbf{a}^*_{13} - q_1 \mathbf{a}^*_{51})] = 0$$

and the analogous relations. Because of $(*)$ and $q_1 q_3 q_5 \neq 0$, the lines S_1, S_3, S_5 have no common normal and are obviously proper. Consequently they determine a hexagon (S_n, $n \bmod 6$) up to orientations. It satisfies the requirement that A_{14}, A_{36}, A_{52} are its altitude lines. Indeed, $(*)$ implies that, for instance, A_{14}, S_3, S_5 have a common normal, and this is S_4.

To obtain a configuration in the plane of the given triangle one has to choose the line O either orthogonal to or lying in the plane. In particular we then have (with more conventional notation):

Let there be given a triangle with improper vertices A^*, B^*, C^*, and let P denote its plane.

If O is any proper or improper point in P different from A^*, B^*, C^*, then the triangle inscribed in $A^* B^* C^*$ whose vertices are the feet of the perpendiculars from O to the sides of $A^* B^* C^*$ has the orthocentre O.

If O is any proper line in P which has no proper or improper point in common with the sides of $A^* B^* C^*$, then the triangle inscribed in $A^* B^* C^*$ whose vertices are the intersections of the sides of $A^* B^* C^*$ and their common normals with O has the orthoaxis O.

It is easily seen that if O is a proper line intersecting two of the sides of $A^* B^* C^*$, one obtains, instead of an inscribed triangle, a self-intersecting quadrangle with two opposite right angles.

Notes to Chapter VI

Schilling [23], [24] § 10–11, discovered that the relations of spherical trigonometry with complex arguments admit of an interpretation as relations concerning a right-angled hexagon in hyperbolic space. He determines the side-lines of the hexagon by the pairs of their ends, and he is aware that these lines have to be oriented. The trigonometric relations he obtains as relations between the cross ratios of the ends. The main tool is the theorem: The product of the squares of the

motions which map S_2 onto itself and S_1 onto S_3, S_4 onto itself and S_3 onto S_5, S_6 onto itself and S_5 onto S_1 is the identity. This is however independent of the orientations, and Schilling's determination of the correct signs in the trigonometric relations seems not convincing.

Right-angled hexagons were also considered by Morley and Morley [17] Chapt. IX. They use also the determination of a line by its ends.

The derivation of the relations by means of trace relations in Section 2 seems to be new.

The relations for plane polygons, obtained in Section 3 by specializing those for a right-angled hexagon, are of course well known. Most of them are to be found in books on non-Euclidean geometry.

To the author's knowledge the determination of a right-angled hexagon by three of its sides (Section 4) has not been dealt with systematically. Schilling [24] § 12, discusses briefly the case where three non-adjacent are given. Most of the cases of plane polygons dealt with in Section 4 are well known.

The notion of amplitude of a non-Euclidean triangle is old; cf. Coolidge [3] Chapt. XIV and the references there. The notion for the right-angled hexagon seems not to have been introduced earlier. The specializations of the relations derived to plane and spherical triangles are to be found in Coolidge [3] Chapt. XIV.

The specializations of the Ceva and Menelaos Theorems for right-angled hexagons, proved in Section 6, to triangles are to be found in Perron [18] § 32.

The bisectors of a right-angled hexagon were considered by Morley and Morley [17] Chapt. IX. Most of the specializations of the results in Section 7 are to be found in various books on non-Euclidean geometry. This does not hold of the inequality (14). Since $\pi - 2\sigma = ar(ABC)$ is the area of the triangle, it states that the radius of the inscribed circle is bounded below by a positive function of the area of the triangle. This has no Euclidean analogue. It was discovered independently by Marden [14] and Sturm and Shinnar [29]. The estimate (14) is not best possible. The right-hand side can be replaced by $\frac{1}{2}\cos\sigma = \frac{1}{2}\sin\frac{1}{2}\,\mathrm{ar}\,(ABC)$. Equality then holds for triangles with two improper vertices. Cf. Beardon [32].

The medians of a right-angled hexagon, as defined in Section 8, seem not to have been considered earlier. The special results for triangles are well known.

That the altitude lines, the common normals of opposite side-lines of a right-angled hexagon, in Euclidean space have a common normal, was discovered 1898 independently by Morley [16] and Petersen [22] (he changed his name to Hjelmslev). That the theorem holds in hyperbolic geometry seems to have been shown for the first time in Morley and Morley [17] Chapt. IX. That it holds for the altitudes of a triangle has been known long before. Various proofs are to be found in books on non-Euclidean geometry. The other results in Section 9 seem to be new.

VII. Points and Planes

VII.1 Point and plane matrices

In analogy with the treatment of lines, we represent a point or plane by a matrix determining the reflection in the point or plane.

The reversal p^* determined by the matrix $\mathbf{p}j$ with $\det \mathbf{p} = 1$ is involutory if and only if $\mathbf{p}\bar{\mathbf{p}} = -\mathbf{1}$ or $\mathbf{p}\bar{\mathbf{p}} = \mathbf{1}$. As shown in IV.3, p^* is a point-reflection in the first case and a plane-reflection in the second. Clearly, any matrix $\lambda \mathbf{p} j$ with $\lambda \in \mathbb{R} \setminus \{0\}$, and no other determines the same reflection. A matrix $\mathbf{p}$ with $\det \mathbf{p} > 0$ will be called a *point matrix* if

$$\mathbf{p}\bar{\mathbf{p}} = -\mathbf{1} \det \mathbf{p}$$

equivalently,

$$\mathbf{p}^{\sim} = -\bar{\mathbf{p}}$$

and $\mathbf{p}$ will be called a *plane matrix* if

$$\mathbf{p}\bar{\mathbf{p}} = \mathbf{1} \det \mathbf{p} \,,$$

equivalently,

$$\mathbf{p}^{\sim} = \bar{\mathbf{p}} \,,$$

Occasionally it will be convenient to admit singular "p-matrices" $\mathbf{p} \neq \mathbf{0}$ satisfying

$$\mathbf{p}\bar{\mathbf{p}} = \mathbf{0} \,.$$

explicitly

$$\begin{aligned} p_{11}\bar{p}_{11} + p_{12}\bar{p}_{21} &= 0 \,, \\ p_{11}\bar{p}_{12} + p_{12}\bar{p}_{22} &= 0 \,, \\ p_{21}\bar{p}_{11} + p_{22}\bar{p}_{21} &= 0 \,, \\ p_{21}\bar{p}_{12} + p_{22}\bar{p}_{22} &= 0 \,. \end{aligned}$$

Suppose first that $p_{21} \neq 0$. Since a non-zero complex factor is of no significance here, we may assume that $p_{21} = 1$. The third equation then yields $p_{22} = -\bar{p}_{11}$ and the first $p_{12} = -p_{11}\bar{p}_{11}$. With the notation $t = p_{11}$ we therefore have

$$\mathbf{p} = \begin{pmatrix} t & -t\bar{t} \\ 1 & -\bar{t} \end{pmatrix}.$$

If $p_{21} = 0$, the first and the last equations yield $p_{11} = p_{22} = 0$. Assuming that $p_{12} = 1$, we have here

$$\mathbf{p} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The mapping determined by such a matrix $\mathbf{p}$ carries every proper or improper point, with the exception of t or ∞, respectively, into t or ∞, respectively. Depending on the context, we shall say that $\mathbf{p}$ determines the *improper point* or *improper plane* t or ∞.

A regular *point matrix* $\mathbf{p}$ satisfies

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^{\sim} = \begin{pmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{pmatrix} = \begin{pmatrix} -\bar{p}_{11} & -\bar{p}_{12} \\ -\bar{p}_{21} & -\bar{p}_{22} \end{pmatrix}.$$

Hence

$$p_{22} = -\bar{p}_{11}, \qquad p_{12}, p_{21} \in \mathbb{R}.$$

Since

$$\det \mathbf{p} = -p_{11}\bar{p}_{11} - p_{12}p_{21} > 0,$$

we have $p_{12}p_{21} < 0$, in particular $p_{21} \neq 0$. The centre $c + \gamma j$ of the point-reflection determined by $\mathbf{p}j$ satisfies

$$p_{11}(\bar{c} + \gamma j) + p_{12} = (c + \gamma j)\,(p_{21}(\bar{c} + \gamma j) - \bar{p}_{11})$$

which may be written

$$\left(c - \frac{p_{11}}{p_{21}}\right)\left(\bar{c} - \frac{\bar{p}_{11}}{\bar{p}_{21}}\right) + \frac{\det \mathbf{p}}{{p_{21}}^2} - \gamma^2 + 2\left(c - \frac{p_{11}}{p_{21}}\right)\gamma j = 0.$$

There is no solution with $\gamma = 0$, hence

$$c = \frac{p_{11}}{p_{21}}, \qquad \gamma^2 = \frac{\det \mathbf{p}}{{p_{21}}^2}.$$

There are two opposite matrices with determinant 1 determining the same point-reflection. We shall call a point matrix $\mathbf{p}$ *normalized* if

$$\det \mathbf{p} = 1, \qquad p_{21} > 0.$$

For a given point $c + \gamma j$ this matrix is

$$\begin{pmatrix} \dfrac{c}{\gamma} & -\dfrac{c\bar{c}+\gamma^2}{\gamma} \\ \dfrac{1}{\gamma} & -\dfrac{\bar{c}}{\gamma} \end{pmatrix}.$$

It has to be checked that a conjugate of a normalized point matrix is normalized, hence, that the element p_{21} of the matrix $\mathbf{p}' = \mathbf{f}\mathbf{p}\bar{\mathbf{f}}^{-1}$ is positive for every $\mathbf{f} \in \mathrm{GL}(2, \mathbb{C})$ (cf. IV 3(4)). It is sufficient to consider

$$\mathbf{f} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in \mathbb{C}, \quad \mathbf{f} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the first case one finds $p'_{21} = p_{21}$ and in the second $p'_{21} = -p_{12}$ which is positive since $-p_{12}p_{21} > 0$. The sign of p_{21} is also preserved under conjugation by reversals. To see this one has only to apply $\mathbf{1}j$ which transforms $\mathbf{p}$ to $\bar{\mathbf{p}}$.

Consider now a regular *plane matrix* $\mathbf{P}$. (To facilitate the reading we shall denote plane matrices by capital letters). Since

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}^{\sim} = \begin{pmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{pmatrix} = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix},$$

we have here

$$P_{22} = \bar{P}_{11}, \qquad P_{12}, P_{21} \in i\,\mathbb{R}.$$

An equation for the mirror of the corresponding plane reflection is

$$P_{11}(\bar{x} + \xi j) + P_{12} = (x + \xi j)(P_{21}(\bar{x} + \xi j) + \bar{P}_{11})$$

which reduces to

$$P_{21}(x\bar{x} + \xi^2) + \bar{P}_{11}x - P_{11}\bar{x} - P_{12} = 0.$$

If $P_{21} = 0$, the mirror is a vertical e-half-plane with normal vector $P_{11}i$. If $P_{21} \neq 0$, the equation may be written

$$\left(x - \frac{P_{11}}{P_{21}}\right)\left(\bar{x} - \frac{\bar{P}_{11}}{\bar{P}_{21}}\right) + \xi^2 + \frac{\det \mathbf{P}}{P_{21}^2} = 0,$$

so the mirror is the e-hemisphere with e-center P_{11}/P_{21} and e-radius $|\det^{1/2}\mathbf{P}/P_{21}|$. If an equation of the mirror is given in the form

$$\alpha(x\bar{x} + \xi^2) - \bar{b}x - b\bar{x} + \gamma = 0, \quad \alpha, \gamma \in \mathbb{R}, \quad b \in \mathbb{C}, \quad b\bar{b} - \alpha\gamma > 0,$$

a corresponding plane matrix is

$$\begin{pmatrix} bi & -\gamma i \\ \alpha i & -\bar{b}i \end{pmatrix}.$$

A proper plane P is determined by two opposite *normalized plane matrices* with determinant 1. With each of them an orientation of P can be associated in a consistent manner. Let $\mathbf{P}$ be one of the normalized matrices determining P. The *orientation* of P associated with $\mathbf{P}$, in the sense that one of the half-spaces bounded by P is considered the positive one, is defined as follows. Let r be a proper point of P and $\mathbf{r}$ the normalized point matrix determining it. Then $\mathbf{r}\bar{\mathbf{P}}$ determines a half-turn, so

$$\mathbf{l} = \mathbf{r}\bar{\mathbf{P}}$$

is a normalized line matrix determining the normal L to P through r with an orientation. The positive half-space is the one into which L points.

It has to be checked that, under a motion f, the positive half-space determined by $\mathbf{P}$ is mapped onto the positive half-space determined by $\mathbf{f}\mathbf{P}\bar{\mathbf{f}}^{-1}$ (cf. IV.3(4)). Because of

$$\mathbf{f}\mathbf{r}\bar{\mathbf{f}}^{-1}\overline{\mathbf{f}\mathbf{P}\bar{\mathbf{f}}^{-1}} = \mathbf{f}\mathbf{l}\mathbf{f}^{-1}$$

and the fact, proved above, that $\mathbf{f}\mathbf{r}\bar{\mathbf{f}}^{-1}$ is a normalized point matrix, this is a consequence of the preservation of the orientation of lines. Under reversals the orientation is reversed since this is the case for lines. Application of motions carrying the plane P and each of the half-spaces bounded by it into themselves, shows that the orientation of P is independent of the choice of the point r in P.

The orientation of a plane P may be read off from the matrix $\mathbf{P}$ as follows. If $P_{21} \neq 0$ and thus P an e-hemisphere, its exterior is the positive or negative half-space according as $P_{21}i < 0$ or $P_{21}i > 0$. If $P_{21} = 0$ and thus P a vertical e-half-plane, the positive half-space is the one into which the vector $P_{11}i$ points.

To see this, observe that for $b \in \mathbb{C}$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & \bar{P}_{11} \end{pmatrix}\begin{pmatrix} 1 & -\bar{b} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} P_{11} + bP_{21} & b\bar{P}_{11} - \bar{b}P_{11} + P_{12} - b\bar{b}P_{21} \\ P_{21} & \bar{P}_{11} - \bar{b}P_{21} \end{pmatrix}.$$

Hence, P_{21} remains unchanged. If $P_{21} \neq 0$, we may choose $b = -P_{11}/P_{21}$ and, because of $\det \mathbf{P} = 1$, assume that

$$\mathbf{P} = \begin{pmatrix} 0 & -1/P_{21} \\ P_{21} & 0 \end{pmatrix}.$$

Then P is the e-hemisphere with centre 0 and radius $1/|P_{21}|$. It contains the point $\mathrm{r} = j/|P_{21}|$ with the normalized point matrix

$$\mathbf{r} = \begin{pmatrix} 0 & -1/|P_{21}| \\ |P_{21}| & 0 \end{pmatrix}.$$

Now

$$\mathbf{r}\bar{\mathbf{P}} = \begin{pmatrix} 0 & -1/|P_{21}| \\ |P_{21}| & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/P_{21} \\ -P_{21} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} P_{21}/|P_{21}| & 0 \\ 0 & |P_{21}|/P_{21} \end{pmatrix}$$

is the line matrix of $[0, \infty]$ if $P_{21}/|P_{21}| = i$ and of $[\infty, 0]$ if $P_{21}/|P_{21}| = -i$ (cf. V. 2). This proves the first statement. If $P_{21} = 0$, we may choose b such that

$$\bar{b}P_{11} - b\bar{P}_{11} = P_{12}.$$

Then $\mathbf{P}$ takes the form

$$\mathbf{P} = \begin{pmatrix} P_{11} & 0 \\ 0 & \bar{P}_{11} \end{pmatrix}, \quad P_{11}\bar{P}_{11} = 1,$$

and P is a vertical e-half-plane having 0 on its boundary. Hence, we may choose $\mathrm{r} = j$ with the normalized point matrix

$$\mathbf{r} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{r}\bar{\mathbf{P}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{P}_{11} & 0 \\ 0 & P_{11} \end{pmatrix} = \begin{pmatrix} 0 & -P_{11} \\ \bar{P}_{11} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -P_{11} \\ 1/P_{11} & 0 \end{pmatrix}$$

is the line matrix determining $[-P_{11}i, P_{11}i]$ (cf. V.2), and this proves the second statement.

VII.2 Incidence and orthogonality

Let L be a line, r a point, and P a plane, all proper and determined by the normalized matrices $\mathbf{l}, \mathbf{r}, \mathbf{P}$, respectively. According to the preceding, L and P are orthogonal, concordantly oriented, and intersect at r if and only if

(1) $\quad \mathbf{l} = \mathbf{r}\bar{\mathbf{P}} = \mathbf{P}\bar{\mathbf{r}}$

which, because of $\mathbf{r} = -\bar{\mathbf{r}}$ and $\mathbf{P} = \bar{\mathbf{P}}$ may also be written

(2) $$\mathbf{P} = -\mathbf{r}\bar{\mathbf{l}} = -\mathbf{l}\mathbf{r}$$

or

(3) $$\mathbf{r} = \mathbf{l}\mathbf{P} = \mathbf{P}\bar{\mathbf{l}}\,.$$

The following simple statements are obvious corollaries hereof in case the elements involved are proper, but they hold also if improper elements occur (cf III.2 for definitions).

A proper or improper point r, *determined by the point matrix* $\mathbf{r}$, *lies in the proper or improper plane* P, *determined by the plane matrix* $\mathbf{P}$, *if and only if*

(4) $$\operatorname{tr}(\mathbf{r}\bar{\mathbf{P}}) = 0\,.$$

If r and P are proper, this is a consequence of (1). If r is improper, it may be assumed to be ∞, so

$$\mathbf{r} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\operatorname{tr}(\mathbf{r}\bar{\mathbf{P}}) = \bar{P}_{21}\,,$$

and this vanishes if and only if P is a vertical e-half-plane or ∞. Hence, if and only if ∞ belongs to the horizon of P or coincides with P. If P is improper and assumed to be ∞, the condition is $r_{21} = 0$ which, because of $\det \mathbf{r} > 0$, implies $r_{11} = 0$ and thus r $= \infty$.

A proper or improper point r, *determined by the point matrix* $\mathbf{r}$, *lies on a proper or improper line* L, *determined by the line matrix* $\mathbf{l}$, *if and only if*

(5) $$\mathbf{l}\mathbf{r} - \mathbf{r}\bar{\mathbf{l}} = \mathbf{0}\,.$$

If r and L are proper, this is a consequence of (2). If r is improper, it may be assumed to be ∞. Then

$$\begin{aligned}\mathbf{l}\mathbf{r} - \mathbf{r}\bar{\mathbf{l}} &= \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & -l_{11} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{l}_{11} & \bar{l}_{12} \\ \bar{l}_{21} & -\bar{l}_{11} \end{pmatrix} \\ &= \begin{pmatrix} -\bar{l}_{21} & l_{11} + \bar{l}_{11} \\ 0 & l_{21} \end{pmatrix}\end{aligned}$$

and thus the condition is $l_{21} = 0, l_{11} \in i\mathbb{R}$, which means that L is a vertical e-half-

line. Hence, it says that r is an end of L. If L is improper and assumed to be ∞, we have to consider

$$(6) \qquad \mathbf{l} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \qquad t \in \mathbb{C}\setminus\{0\},$$

since the condition changes if $\mathbf{l}$ is multiplied by a non-zero factor. It is therefore

$$\mathbf{lr} - \mathbf{r\bar{l}} = \begin{pmatrix} t\,r_{21} & -\bar{t}\,r_{11} - t\,\bar{r}_{11} \\ 0 & -t\,r_{21} \end{pmatrix} = \mathbf{0},$$

hence $r_{21} = 0$ which because of $\det \mathbf{r} \geq 0$, implies $r_{11} = 0$ and thus $\mathrm{r} = \infty$.

A proper or improper line L, determined by the line matrix $\mathbf{l}$, is orthogonal to the plane P, determined by the plane matrix $\mathbf{P}$, if and only if

$$(7) \qquad \mathbf{lP} - \mathbf{P\bar{l}} = \mathbf{0}.$$

If L and P are proper, this is a consequence of (3). If P is improper and assumed to be ∞, the condition is seen as above that L is a vertical e-half-line. If L is improper and $\mathbf{l}$ chosen to be (6), the condition amounts to $P_{21} = 0$. Hence, it says that P is a vertical e-half-plane, and this is the statement.

Further we observe:

A proper or improper line L, determined by the line matrix $\mathbf{l}$, lies in the proper plane P, determined by the plane matrix $\mathbf{P}$, if and only if

$$(8) \qquad \mathbf{lP} + \mathbf{P\bar{l}} = \mathbf{0}.$$

If L is proper, this follows from the fact that $L \subset P$ if and only if the reversal determined by $\mathbf{lP}j$ is a plane-reflection, that is $(\mathbf{lP})^\sim = -\mathbf{\bar{P}l} = \overline{\mathbf{lP}}$. If L is improper, we may assume $\mathbf{l}$ to be the matrix in (6). Then (8) is satisfied precisely for $P_{21} = 0$ and an arbitrary P_{11} (depending on t), that is, for all vertical e-half-planes P.

Consider now two proper oriented planes P and Q determined by the normalized matrices $\mathbf{P}$ and $\mathbf{Q}$. *They intersect orthogonally if and only if the motion determined by $\mathbf{Q\bar{P}}$ is a half-turn, thus $\mathbf{Q\bar{P}}$ a line matrix. Hence the condition is*

$$(9) \qquad \operatorname{tr}(\mathbf{Q\bar{P}}) = 0$$

or, equivalently,

$$(10) \qquad \mathbf{Q\bar{P}} + \mathbf{P\bar{Q}} = \mathbf{0}.$$

Suppose this satisfied. Then $\mathbf{l} = \mathbf{Q\bar{P}}$ determines the line $L = P \cap Q$ with a certain orientation. We claim that the rotation through $\pi/2$ about L in the positive sense maps the positive half-space of P onto the positive half-space of Q. To

see this, we may assume that P is the vertical e-half-plane bounded by the real axis and oriented in accordance with the imaginary axis, and that Q is the vertical e-half-plane bounded by the imaginary axis and oriented opposite the real axis. Then

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and

$$\mathbf{l} = \mathbf{Q}\bar{\mathbf{P}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which determines $[0, \infty]$ as claimed.

Consider now a reversal f^* determined by $\mathbf{f}j$ with $\det \mathbf{f} = 1$. Assume that f^* is not a point-reflection. *We claim that* $\mathbf{f} + \bar{\mathbf{f}}^{\sim}$ *is a plane matrix determining the mirror of* f^*.

To prove this we use that f^* is the product $t^* \circ s^* \circ r^*$ of three plane-reflections such that the mirror of t^* is orthogonal to those of others and thus the mirror of f^* (cf. IV.3). Since f^* is not a point-reflection, the mirrors of r^* and s^* do not intersect orthogonally. With normalized matrices $\mathbf{R}, \mathbf{S}, \mathbf{T}$ determining the reflections we have

$$\mathbf{f} = \mathbf{T}j\mathbf{S}j\mathbf{R} = -\mathbf{T}\bar{\mathbf{S}}\mathbf{R}$$

hence, applying (10) to $\mathbf{S}, \mathbf{T}$ and $\mathbf{R}, \mathbf{T}$,

$$\begin{aligned} \mathbf{f} + \bar{\mathbf{f}}^{\sim} &= -\mathbf{T}\bar{\mathbf{S}}\mathbf{R} - \mathbf{R}\bar{\mathbf{S}}\mathbf{T} = -\mathbf{S}\bar{\mathbf{R}}\mathbf{T} - \mathbf{R}\bar{\mathbf{S}}\mathbf{T} \\ &= -(\mathbf{S}\bar{\mathbf{R}} + \mathbf{R}\bar{\mathbf{S}})\mathbf{T}\,. \end{aligned}$$

Now

$$\mathbf{S}\bar{\mathbf{R}} + \mathbf{R}\bar{\mathbf{S}} = \mathbf{S}\bar{\mathbf{R}} + (\mathbf{S}\bar{\mathbf{R}})^{\sim} = \mathbf{1}\,\mathrm{tr}(\mathbf{S}\bar{\mathbf{R}})$$

and $\mathrm{tr}(\mathbf{S}\bar{\mathbf{R}}) \neq 0$ since the mirrors of r^* and s^* do not intersect orthogonally. Further $\mathrm{tr}(\mathbf{S}\bar{\mathbf{R}})$ is real since $s^* \circ r^*$ is not a skrew motion. Hence, up to a non-zero real factor $\mathbf{f} + \bar{\mathbf{f}}^{\sim}$ equals $\mathbf{T}$, and this is the statement.

Consider now the matrix $\mathbf{f} - \bar{\mathbf{f}}^{\sim}$. We have

$$\begin{aligned} &(\mathbf{f} - \bar{\mathbf{f}}^{\sim})^{\sim} = \mathbf{f}^{\sim} - \bar{\mathbf{f}} = -\overline{(\mathbf{f} - \bar{\mathbf{f}}^{\sim})}, \\ &\det(\mathbf{f} - \bar{\mathbf{f}}^{\sim}) = 2 - \mathrm{tr}(\mathbf{f}\bar{\mathbf{f}})\,. \end{aligned}$$

Further we note that

$$(11) \qquad (\mathbf{f} - \bar{\mathbf{f}}^{\sim})\overline{(\mathbf{f} + \bar{\mathbf{f}}^{\sim})} = (\mathbf{f} - \bar{\mathbf{f}}^{\sim})(\bar{\mathbf{f}} + \mathbf{f}^{\sim}) = \mathbf{f}\bar{\mathbf{f}} - \bar{\mathbf{f}}^{\sim}\mathbf{f}^{\sim} = \mathbf{f}\bar{\mathbf{f}} - (\mathbf{f}\bar{\mathbf{f}})^{\sim}$$

is a line matrix determining the common axis of the motion f^{*2} and the reversal f^*, excluding here the case of a plane-reflection f^*. We have to distinguish three cases. Suppose $\mathrm{tr}(\mathbf{f}\bar{\mathbf{f}}) < 2$. Then f^* is a rotary reflection and $\mathbf{f} - \bar{\mathbf{f}}^{\sim}$ a point matrix.

Though the matrices in (11) are not normalized, we may use (1) to conclude that $\mathbf{f}-\bar{\mathbf{f}}^{\sim}$ determines the point of intersection of the axis and the mirror of f^*, that is, the fixed point of f^*. (This holds obviously also if f^* is a point-reflection since then $\mathbf{f}-\bar{\mathbf{f}}^{\sim}=2\mathbf{f}$). Suppose next that $\operatorname{tr}(\mathbf{f}\bar{\mathbf{f}})=2$. Then f^* is either a plane-reflection and then $\mathbf{f}-\bar{\mathbf{f}}^{\sim}=\mathbf{0}$ or it is a parallel reflection. In the latter case $\mathbf{f}-\bar{\mathbf{f}}^{\sim}$ is singular but not $\mathbf{0}$, and it determines the improper fixed point of f^*. This is most easily seen by assuming.

$$\mathbf{f}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

(cf. IV.3). Then

$$\mathbf{f}-\bar{\mathbf{f}}^{\sim}=\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}$$

which determines the fixed point ∞ of f^*. Finally suppose $\operatorname{tr}(\mathbf{f}\bar{\mathbf{f}})>2$. Then f^* is a glide reflection and $(\mathbf{f}-\bar{\mathbf{f}}^{\sim})i$ is a plane matrix. Using (9), we infer from (11) (multiplied by i) that $\mathbf{f}-\bar{\mathbf{f}}^{\sim}$ *determines the plane through the axis of f^* and orthogonal to the mirror of f^*.*

VII.3 Distances and angles

In this section points, lines and planes are assumed to be proper if not stated otherwise.

Let r and s be points determined by the normalized matrices $\mathbf{r}$ and $\mathbf{s}$. Further, let L, determined by the normalized matrix $\mathbf{l}$, be an oriented line passing through r and s. The distance from r to s provided with a sign in accordance with the orientation of L will be denoted by $\delta(\mathrm{r},\mathrm{s})$. Then

$$\text{(1)}\qquad \cosh\delta(\mathrm{r},\mathrm{s})=\frac{1}{2}\operatorname{tr}(\mathbf{s}\bar{\mathbf{r}})=-\frac{1}{2}\operatorname{tr}(\mathbf{r}\bar{\mathbf{s}}),$$

$$\text{(2)}\qquad \sinh\delta(\mathrm{r},\mathrm{s})=\frac{i}{2}\operatorname{tr}(\mathbf{s}\bar{\mathbf{r}}\mathbf{l})=\frac{i}{2}\operatorname{tr}(\mathbf{s}\bar{\mathbf{l}}\mathbf{r})=\frac{i}{2}\operatorname{tr}(\mathbf{l}\mathbf{s}\bar{\mathbf{r}}),$$

To see this, one may assume that $\mathrm{r}=j$, $\mathrm{s}=\sigma j$, $\sigma>0$, and $L=[0,\infty]$. Then

$$\mathbf{r}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\qquad \mathbf{s}=\begin{pmatrix}0 & -\sigma\\ 1/\sigma & 0\end{pmatrix},\qquad \mathbf{l}=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix},$$

hence

$$\operatorname{tr}(\mathbf{s}\bar{\mathbf{r}})=-\sigma-1/\sigma,\quad \operatorname{tr}(\mathbf{s}\bar{\mathbf{r}}\mathbf{l})=-\sigma i+i/\sigma.$$

Since $\sigma=\exp\delta(\mathrm{r},\mathrm{s})$, the statements follow.

Given $\mathbf{r}$ and $\mathbf{s}$, $\mathbf{r} \neq \mathbf{s}$, a line matrix determining L is $\mathbf{s}\bar{\mathbf{r}} - (\mathbf{s}\bar{\mathbf{r}})^{\sim} = \mathbf{s}\bar{\mathbf{r}} - \mathbf{r}\bar{\mathbf{s}}$ since L is the axis of the translation determined by $\mathbf{s}\bar{\mathbf{r}}$ (cf. V.1). By (1)

$$\det(\mathbf{s}\bar{\mathbf{r}} - \mathbf{r}\bar{\mathbf{s}}) = 2 - \operatorname{tr}(\mathbf{s}\bar{\mathbf{r}})^2 = 4 - \operatorname{tr}^2(\mathbf{s}\bar{\mathbf{r}}) = -4\sinh^2\delta(\mathrm{r},\mathrm{s}).$$

Hence, a normalized line matrix for L is

$$\text{(3)} \qquad \mathbf{l} = \frac{1}{2i\sinh\delta(\mathrm{r},\mathrm{s})}(\mathbf{s}\bar{\mathbf{r}} - \mathbf{r}\bar{\mathbf{s}}).$$

The orientation determined by this matrix agrees with the sign of $\delta(\mathrm{r},\mathrm{s})$. This may be checked by specializing as above.

Let now P be an oriented plane, s a point, and N the normal to P through s oriented in accordance with P. Denote the normalized matrices determining P, s and N by $\mathbf{P}$, $\mathbf{s}$, and $\mathbf{n}$, respectively. Then we have for the distance $\delta(P,\mathrm{s})$ from P to s provided with a sign in accordance with the orientation of P (cf. VII,2(2),(3)).

$$\text{(4)} \qquad \cosh\delta(P,\mathrm{s}) = -\frac{1}{2}\operatorname{tr}(\mathbf{s}\bar{\mathbf{n}}\bar{\mathbf{P}}) = -\frac{1}{2}\operatorname{tr}(\mathbf{n}\mathbf{s}\bar{\mathbf{P}}) = \frac{1}{2}\operatorname{tr}(\mathbf{s}\bar{\mathbf{P}}\mathbf{n}),$$

$$\text{(5)} \qquad \sinh\delta(P,\mathrm{s}) = -\frac{i}{2}\operatorname{tr}(\mathbf{s}\bar{\mathbf{P}}) = -\frac{i}{2}\operatorname{tr}(\mathbf{P}\bar{\mathbf{s}}).$$

According to VII.1(4), $\mathbf{r} = \mathbf{n}\mathbf{P}$ is the normalized point matrix determining $\mathrm{r} = N \cap P$. Replacing in the expressions on the right-hand sides of (1) and (2) $\mathbf{r}$ by $\mathbf{n}\mathbf{P} = \mathbf{P}\bar{\mathbf{n}}$ and $\mathbf{l}$ by $\mathbf{n}$, and observing that $\delta(P,\mathrm{s}) = \delta(\mathrm{r},\mathrm{s})$, one sees that (4) and (5) follow from (1) and (2).

If $\mathbf{P}$ and $\mathbf{s}$ are given, a line matrix determining N is $\mathbf{P}\bar{\mathbf{s}} - (\mathbf{P}\bar{\mathbf{s}})^{\sim} = \mathbf{P}\bar{\mathbf{s}} + \mathbf{s}\bar{\mathbf{P}}$, since N is the axis of the skrew motion determined by $\mathbf{P}\bar{\mathbf{s}}$. Now, by (5)

$$\det(\mathbf{P}\bar{\mathbf{s}} + \mathbf{s}\bar{\mathbf{P}}) = 2 - \operatorname{tr}(\mathbf{P}\bar{\mathbf{s}})^2 = 4 - \operatorname{tr}^2(\mathbf{P}\bar{\mathbf{s}}) = 4\cosh^2\delta(P,\mathrm{s}).$$

Hence, a normalized line matrix for N is

$$\text{(6)} \qquad \mathbf{n} = \frac{1}{2\cosh\delta(P,\mathrm{s})}(\mathbf{P}\bar{\mathbf{s}} + \mathbf{s}\bar{\mathbf{P}}).$$

The orientation of N determined by it agrees with that of P. This may easily be checked by assuming P to be the unit e-hemisphere with center 0, oriented towards its exterior and that $\mathrm{s} = \sigma j$, $\sigma > 0$. Then one has $N = [0, \infty]$ since

$$\mathbf{P} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0 & -\sigma \\ 1/\sigma & 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The normalized point matrix determining $\mathrm{r} = N \cap P$ is (cf. VII.2(3))

$$\text{(7)} \qquad \mathbf{r} = \mathbf{n}\mathbf{P} = \frac{1}{2\cosh\delta(P,\mathrm{s})}(\mathbf{P}\bar{\mathbf{s}}\mathbf{P} + \mathbf{s}).$$

Consider now ultraparallel oriented planes P and Q, and let N be their common normal oriented in accordance with P. The normalized matrices determining them are denoted by $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{n}$. For the distance $\delta(P,Q)$ from P to Q provided with a sign in accordance with the orientation of N we then have (cf. VII.2(7))

$$\cosh\delta(P,Q) = \pm\frac{1}{2}\operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}}) = \pm\frac{1}{2}\operatorname{tr}(\mathbf{P}\bar{\mathbf{Q}}), \tag{8}$$

$$\sinh\delta(P,Q) = \mp\frac{i}{2}\operatorname{tr}(\mathbf{Q}\bar{\mathbf{n}}\bar{\mathbf{P}}) = \mp\frac{i}{2}\operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}}\mathbf{n}) = \mp\frac{i}{2}\operatorname{tr}(\mathbf{n}\mathbf{Q}\bar{\mathbf{P}}) \tag{9}$$

with the upper signs if P and Q are concordantly oriented otherwise with the lower signs.

If P, Q and thus N are concordantly oriented, $\mathbf{s} = \mathbf{nQ}$ is the normalized matrix determining the point $\mathrm{s} = N \cap Q$. Since $\delta(P,Q) = \delta(P,\mathrm{s})$, (8) and (9) are obtained from (4) and (5) by replacing $\mathbf{s}$ on the right-hand sides of these by $\mathbf{nQ}$. If P and Q are oppositely oriented, the argument applies with $-\mathbf{Q}$ instead of $\mathbf{Q}$.

Since N is the axis of the translation determined by $\mathbf{Q}\bar{\mathbf{P}}$, a line matrix for N is $\mathbf{Q}\bar{\mathbf{P}} - (\mathbf{Q}\bar{\mathbf{P}})^{\sim} = \mathbf{Q}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{Q}}$. Its determinant is

$$\det(\mathbf{Q}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{Q}}) = 2 - \operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}})^2 = 4 - \operatorname{tr}^2(\mathbf{Q}\bar{\mathbf{P}}) = -4\sinh^2\delta(P,Q)$$

and hence the normalized line matrix

$$\mathbf{n} = \pm\frac{i}{2\sinh\delta(P,Q)}(\mathbf{Q}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{Q}}) \tag{10}$$

with the upper or lower sign according as P and Q are concordantly or oppositely oriented. That $\mathbf{n}$ determines the correct orientation of N may be checked by assuming P and Q to be e-hemispheres with center 0 and e-radii 1 and $\varrho > 0$, both oriented towards their exteriors. Then one has indeed

$$\mathbf{P} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0 & i\varrho \\ i/\varrho & 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Normalized point matrices determining $\mathrm{r} = N \cap P$ and $\mathrm{s} = N \cap Q$ are

$$\mathbf{r} = \mathbf{nP} = \pm\frac{i}{2\sinh\delta(P,Q)}(\mathbf{Q} - \mathbf{P}\bar{\mathbf{Q}}\mathbf{P}), \tag{11}$$

$$\mathbf{s} = \mathbf{nQ} = \frac{i}{2\sinh\delta(P,Q)}(\mathbf{Q}\bar{\mathbf{P}}\mathbf{Q} - \mathbf{P}) \tag{12}$$

again with upper or lower signs according as P and Q are concordantly or oppositely oriented.

If the oriented planes P and Q are parallel, one has, as to be expected,

$$\operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}}) = \operatorname{tr}(\mathbf{P}\bar{\mathbf{Q}}) = \pm 2 \tag{13}$$

with the upper sign if P and Q are concordantly oriented, otherwise with the lower sign.

To see this one may assume that P and Q are the vertical e-half-planes with the real axis and its parallel through i as horizons. Then (13) is obvious since the normalized matrices determining P and Q, both oriented in accordance with the imaginary axis, are

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}.$$

The matrix $\mathbf{Q}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{Q}}$, which is singular here, determines the improper common normal of P and Q, that is, the point of contact of their horizons.

Let now P and Q be oriented planes which intersect in a line M, and assume M to be oriented. The angle $\varphi(P, Q)$ is by definition the angle from the positive normal of P to the positive normal of Q taken in the positive sense about M. If $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{m}$ are the normalized matrices determining P, Q, M, respectively, then

$$\cos\varphi(P, Q) = \frac{1}{2}\operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}}) = \frac{1}{2}\operatorname{tr}(\mathbf{P}\bar{\mathbf{Q}}), \tag{14}$$

$$\sin\varphi(P, Q) = -\frac{1}{2}\operatorname{tr}(\mathbf{m}\mathbf{Q}\bar{\mathbf{P}}) = \frac{1}{2}\operatorname{tr}(\mathbf{Q}\bar{\mathbf{m}}\bar{\mathbf{P}}) = -\operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}}\mathbf{m}). \tag{15}$$

To prove this, we may assume that M is $[0, \infty]$ and P is the vertical e-half-plane bounded by the real axis and oriented in accordance with the imaginary axis. Then Q is the e-half-plane obtained, orientation included by rotating P about $[0, \infty]$ through the angle $\varphi(P, Q)$ in the positive sense. Then

$$\mathbf{m} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} \exp(i\,\varphi(P, Q)) & 0 \\ 0 & \exp(-i\,\varphi(P, Q)) \end{pmatrix}$$

and the statements are easily checked.

Since $\mathbf{Q}\bar{\mathbf{P}}$ determines a rotation about M, a line matrix determining M is $\mathbf{Q}\bar{\mathbf{P}} - (\mathbf{Q}\bar{\mathbf{P}})^{\sim} = \mathbf{Q}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{Q}}$. Since

$$\det(\mathbf{Q}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{Q}}) = 2 - \operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}})^2 = 4 - \operatorname{tr}^2(\mathbf{Q}\mathbf{P}) = 4\sin^2\varphi(P, Q),$$

the normalized line matrix for M may be written

$$\mathbf{m} = \frac{1}{2\sin\varphi(P, Q)}(\mathbf{Q}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{Q}}). \tag{16}$$

Consider now a point r and an oriented line L, not through r, determined by the normalized matrices $\mathbf{r}$ and $\mathbf{l}$, respectively. Let $\delta(\mathrm{r}, L)$ denote the positive distance from r to L. Then

$$(17) \qquad \cosh 2\delta(\mathrm{r}, L) = \frac{1}{2}\operatorname{tr}(\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}) = \frac{1}{2}\operatorname{tr}(\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}\mathbf{l}).$$

Since $\mathbf{l}\mathbf{r}j$ determines a glide reflection through the distance $2\delta(\mathrm{r}, L)$, this is a consequence of IV.3(6).

A line matrix determining the normal N of L through r is $\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}} - (\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}})^{\sim} = \mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}} - \mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}\mathbf{l}$. Its determinant is

$$\begin{aligned}\det(\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}} - \mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}\mathbf{l}) &= 2 - \operatorname{tr}(\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}})^2 \\ &= 4 - \operatorname{tr}^2(\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}) = 4\sinh^2 2\delta(L, \mathrm{r}).\end{aligned}$$

Hence, a normalized line matrix for N is

$$(18) \qquad \mathbf{n} = \frac{i}{2\sinh 2\delta(\mathrm{r}, L)}(\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}} - \mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}\mathbf{l}).$$

The orientation of N determined by this matrix is from r towards L. To see this one may assume that $L = [-1, 1]$ and $\mathrm{r} = \sigma j$ with $\sigma > 1$. Then

$$\mathbf{l} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \mathbf{r} = \begin{pmatrix} 0 & -\sigma \\ 1/\sigma & 0 \end{pmatrix},$$

and one finds

$$\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}} - \mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}\mathbf{l} = -i\left(\sigma^2 - \frac{1}{\sigma^2}\right)\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Hence, $N = [\infty, 0]$ since $\sigma > 1$.

The mirror of the glide reflection determined by $\mathbf{l}\mathbf{r}j$ is the plane P through r and orthogonal to L. As shown at the end of the preceding section, a plane matrix determining P is $\mathbf{l}\mathbf{r} + (\bar{\mathbf{l}}\bar{\mathbf{r}})^{\sim} = \mathbf{l}\mathbf{r} + \mathbf{r}\bar{\mathbf{l}}$ with determinant

$$\begin{aligned}\det(\mathbf{l}\mathbf{r} + \mathbf{r}\bar{\mathbf{l}}) &= 2 + \operatorname{tr}(\mathbf{l}\mathbf{r}\bar{\mathbf{l}}\bar{\mathbf{r}}) \\ &= 2 + 2\cosh 2\delta(\mathrm{r}, L) = 4\cosh^2\delta(\mathrm{r}, L).\end{aligned}$$

Hence, a normalized matrix for P is

$$(19) \qquad \mathbf{P} = \frac{-1}{2\cosh\delta(\mathrm{r}, L)}(\mathbf{l}\mathbf{r} + \mathbf{r}\bar{\mathbf{l}}).$$

The orientation of P which it determines agrees with that of L. This may be checked by specializing to $L = [-1, 1]$ and $\mathrm{r} = \sigma j$ as above.

A plane matrix for the plane Q spanned by r and L is as shown at the end of the preceding section $(\mathbf{lr}-\mathbf{r\bar{l}})\,i$ with determinant

$$\begin{aligned}\det i(\mathbf{lr}-\mathbf{r\bar{l}}) &= -2+\operatorname{tr}(\mathbf{lr\bar{l}\bar{r}})\\ &= -2+2\cosh 2\,\delta(\mathrm{r},L) = 4\sinh^2\delta(\mathrm{r},L)\,.\end{aligned}$$

Hence a normalized matrix for Q is

$$(20)\qquad \mathbf{Q} = \frac{i}{2\sinh\delta(\mathrm{r},L)}(\mathbf{lr}-\mathbf{r\bar{l}})\,.$$

The orientation of Q it determines is such that N, L and the positive normal of Q taken in this order form a positive (right-handed) frame. This can also be seen by specializing as above.

Applying VII.2(3) to L and P we obtain the normalized point matrix

$$(21)\qquad \mathbf{s} = \mathbf{lP} = \frac{1}{2\cosh\delta(\mathrm{r},L)}(\mathbf{r}-\mathbf{lr\bar{l}})$$

for the point $\mathrm{s} = N\cap L$.

Finally we consider a plane P determined by the normalized matrix $\mathbf{P}$ and a line L, not normal to P, determined by the normalized matrix $\mathbf{l}$. If P and L are ultraparallel, let $\delta(P,L)$ denote the positive distance from P to L. If P and L intersect, let $\varphi(P,L)\in \left]\!\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\!\right[$ denote the acute angle between P and L taken positive or negative according as P and L are concordantly or oppositely oriented. If P and L are parallel, let $\delta(P,L)=\varphi(P,L)=0$. Since $\mathbf{lP}j$ determines a glide reflection in the first case and a rotary reflection in the second, we have (cf. IV.3(6), (8))

$$(22)\qquad \left.\begin{matrix}\cosh 2\,\delta(P,L)\\ \cos 2\,\varphi(P,L)\end{matrix}\right\} = \frac{1}{2}\operatorname{tr}(\mathbf{lP\bar{l}\bar{P}}) = \frac{1}{2}\operatorname{tr}(\mathbf{P\bar{l}\bar{P}l})\,,$$

respectively. In the third case, where $\mathbf{lP}j$ determines a parallel reflection, both equations hold.

A matrix for the mirror of the reversal $\mathbf{lP}j$, that is, the plane Q through L orthogonal to P, is (cf. VII.2) $\mathbf{lP}+(\bar{\mathbf{l}}\bar{\mathbf{P}})^{\sim} = \mathbf{lP}-\mathbf{P\bar{l}}$. Since

$$\det(\mathbf{lP}-\mathbf{P\bar{l}}) = 2+\operatorname{tr}(\mathbf{lP\bar{l}\bar{P}}) = \begin{cases}4\cosh^2\delta(P,L)\\ 4\cos^2\varphi(P,L)\end{cases},$$

a normalized plane matrix determining Q is

$$
(23) \qquad \mathbf{Q} = \begin{cases} \dfrac{1}{2\cosh\delta(P,L)}(\mathbf{lP} - \mathbf{P}\bar{\mathbf{l}}) \\[2ex] \dfrac{1}{2\cos\varphi(P,L)}(\mathbf{lP} - \mathbf{P}\bar{\mathbf{l}}) \end{cases} .
$$

A normalized line matrix determining the orthogonal projection $M = P \cap Q$ of L onto P is

$$
(24) \qquad \mathbf{m} = \mathbf{Q}\bar{\mathbf{P}} = \begin{cases} \dfrac{1}{2\cosh\delta(P,L)}(\mathbf{l} - \mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}) \\[2ex] \dfrac{1}{2\cos\varphi(P,L)}(\mathbf{l} - \mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}) \end{cases} .
$$

If P and L are ultraparallel, $(\mathbf{lP} - (\bar{\mathbf{l}}\bar{\mathbf{P}})^{\sim})i = (\mathbf{lP} + \mathbf{P}\bar{\mathbf{l}})i$ is a plane matrix determining, as shown in the preceding section, the plane through the axis of the glide reflection $\mathbf{lP}j$ and orthogonal to its mirror, that is, the common normal plane R of P and L. Since

$$
\det i(\mathbf{lP} - \mathbf{P}\bar{\mathbf{l}}) = -2 + \operatorname{tr}(\mathbf{lP}\bar{\mathbf{l}}\bar{\mathbf{P}}) = 4\sinh^2\delta(P,L),
$$

a normalized matrix for R is

$$
(25) \qquad \mathbf{R} = \frac{i}{2\sinh\delta(P,L)}(\mathbf{lP} + \mathbf{P}\bar{\mathbf{l}}).
$$

If P and L are parallel, $\mathbf{lP} + \mathbf{P}\bar{\mathbf{l}}$ is singular and determines the common improper point of P and L. If P and L intersect, $\mathbf{lP} + \mathbf{P}\bar{\mathbf{l}}$ is a point matrix for the fixed point of the rotary reflection $\mathbf{lP}j$, that is the point r of intersection of P and L (cf. VII.2). Hence we have

$$
\det(\mathbf{lP} + \mathbf{P}\bar{\mathbf{l}}) = 2 - \operatorname{tr}(\mathbf{lP}\bar{\mathbf{l}}\bar{\mathbf{P}}) = 4\sin^2\varphi(P,L),
$$

and we claim that

$$
(26) \qquad \mathbf{r} = \frac{1}{2\sin\varphi(P,L)}(\mathbf{lP} + \mathbf{P}\bar{\mathbf{l}})
$$

is the normalized point matrix determining r, that is, its element in the first column and the second row is positive. Observing that change of the orientation of L implies change of the sign of $\varphi(P, L)$, one may check the statement assuming that P is the unit e-hemisphere oriented towards its exterior and that $L = [u, \infty]$ with $|u| < 1$. Then $\varphi > 0$ and

$$
\mathbf{P} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{l} = \begin{pmatrix} i & -2ui \\ 0 & -i \end{pmatrix}, \quad \mathbf{lP} + \mathbf{P}\bar{\mathbf{l}} = \begin{pmatrix} 2u & -2 \\ 2 & -2u \end{pmatrix}.
$$

Assume now that L is neither orthogonal to nor contained in P. Then the reversal $\mathbf{l}\mathbf{P}j$ has an axis determined by

$$\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}} - (\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}})^{\sim} = \mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}\mathbf{l}$$

with

$$\det(\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}\mathbf{l}) = 2 - \operatorname{tr}(\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}})^2 = 4 - \operatorname{tr}^2(\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}).$$

If P and L are ultraparallel, N is the common normal of P and L. The determinant equals $-4\sinh^2 2\,\delta(P,L)$. Hence, a normalized line matrix for N is

$$\mathbf{n} = \frac{i}{2\sinh 2\,\delta(P,L)}(\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}\mathbf{l}). \tag{27}$$

If P and L intersect, N is the line in P which intersects L orthogonally. The determinant equals $4\sin^2 2\,\varphi(P,L)$. Hence, a normalized line matrix for N is

$$\mathbf{n} = \frac{1}{2\sin 2\,\varphi(P,L)}(\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}\mathbf{l}). \tag{28}$$

If P and L are parallel, $\mathbf{l}\mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}} - \mathbf{P}\bar{\mathbf{l}}\bar{\mathbf{P}}\mathbf{l}$ is singular and determines the improper line represented by the common improper point of P and L.

VII.4 Pencils of points and planes

Let p and q be distinct proper points and L the line joining them oriented from p towards q, and let $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{l}$ be the normalized matrices determining them. These satisfy (cf. VII.2(5))

$$\mathbf{l}\mathbf{p} - \mathbf{p}\bar{\mathbf{l}} = \mathbf{0}, \qquad \mathbf{l}\mathbf{q} - \mathbf{q}\bar{\mathbf{l}} = \mathbf{0}.$$

For $\lambda, \mu \in \mathbb{R}$, $(\lambda,\mu) \neq (0,0)$, consider the matrix

$$\mathbf{r} = \lambda\mathbf{p} + \mu\mathbf{q}.$$

Obviously,

$$\mathbf{l}\mathbf{r} - \mathbf{r}\bar{\mathbf{l}} = \mathbf{0}$$

and $\mathbf{r}^{\sim} = -\bar{\mathbf{r}}$. Hence, $\mathbf{r}$ is a point matrix determining a point r of L, provided $\det\mathbf{r} \geq 0$. Now (cf. VII.3(1))

$$\begin{aligned}\det\mathbf{r} &= \det(\lambda\mathbf{p} + \mu\mathbf{q}) = \lambda^2 + \mu^2 + \lambda\mu\operatorname{tr}(\mathbf{p}\mathbf{q}^{\sim})\\ &= \lambda^2 + \mu^2 - \lambda\mu\operatorname{tr}(\mathbf{p}\bar{\mathbf{q}}) = \lambda^2 + \mu^2 + 2\lambda\mu\cosh\delta\\ &= (\lambda + \mu\exp\delta)(\lambda + \mu\exp(-\delta)),\end{aligned}$$

where $\delta = \delta(\mathrm{p}, \mathrm{q})$ denotes the distance from p to q. Hence, $\det \mathbf{r} > 0$ if and only if $\lambda = 0$ or

$$(1) \qquad \frac{\mu}{\lambda} < -\exp\delta \quad \text{or} \quad \frac{\mu}{\lambda} > -\exp(-\delta).$$

Since the ends of L correspond to $\mu/\lambda = -\exp\delta$ and $\mu/\lambda = -\exp(-\delta)$ and the point q to $\lambda = 0$, thus $\mu/\lambda = \pm\infty$, all other points of L are obtained for the values of μ/λ satisfying (1). We have

$$(2) \qquad -\frac{\mu}{\lambda} = \frac{\sinh\delta(\mathrm{p},\mathrm{r})}{\sinh\delta(\mathrm{q},\mathrm{r})}.$$

Indeed, application of VII.3(2) with the normalized matrix $\mathbf{r}/(\det\mathbf{r})^{1/2}$ shows that the right-hand side equals

$$\frac{\operatorname{tr}(\mathbf{r}\bar{\mathbf{p}}\mathbf{l})}{\operatorname{tr}(\mathbf{r}\bar{\mathbf{q}}\mathbf{l})} = \frac{\operatorname{tr}[(\lambda\mathbf{p}+\mu\mathbf{q})\bar{\mathbf{p}}\mathbf{l}]}{\operatorname{tr}[(\lambda\mathbf{p}+\mu\mathbf{q})\bar{\mathbf{q}}\mathbf{l}]} = \frac{\mu\operatorname{tr}(\mathbf{q}\bar{\mathbf{p}}\mathbf{l})}{\lambda\operatorname{tr}(\mathbf{p}\bar{\mathbf{q}}\mathbf{l})} = -\frac{\mu}{\lambda},$$

where $\mathbf{p}\bar{\mathbf{p}} = -\mathbf{1}$, $\mathbf{q}\bar{\mathbf{q}} = -\mathbf{1}$, $\operatorname{tr}\mathbf{l} = 0$ and $(\mathbf{q}\bar{\mathbf{p}}\mathbf{l})^{\sim} = -\mathbf{l}\mathbf{p}\bar{\mathbf{q}}$ are used.

Suppose now that $\det\mathbf{r} < 0$ and thus

$$(3) \qquad -\exp\delta < \frac{\mu}{\lambda} < -\exp(-\delta).$$

Then $\mathbf{R} = i\mathbf{r}$ is a plane matrix. It satisfies

$$\mathbf{l} - \mathbf{R}\bar{\mathbf{l}} = \mathbf{0},$$

so the plane R it determines is orthogonal to L (cf. VII.1(6)). The whole hyperbolic pencil of such planes (cf. III.2) is obtained for the values of μ/λ satisfying (3) because those corresponding to $\mu/\lambda = -\exp\delta$ and $\mu/\lambda = -\exp(-\delta)$ are the improper ones coinciding with the ends of L. Application of VII.2(4) yields

$$(4) \qquad -\frac{\mu}{\lambda} = \frac{\cosh\delta(R,\mathrm{p})}{\cosh\delta(R,\mathrm{q})}.$$

Consider now two proper ultraparallel planes P and Q determined by normalized matrices $\mathbf{P}$ and $\mathbf{Q}$. We assume that they are concordantly oriented. It is easily seen which changes have to be made in the following if they are oppositely oriented. Let L denote the common normal, oriented in accordance with the planes, and let $\mathbf{l}$ be the normalized matrix determining it. We then have (cf. VII.2(7))

$$\mathbf{l}\mathbf{P} - \mathbf{P}\bar{\mathbf{l}} = \mathbf{0}, \qquad \mathbf{l}\mathbf{Q} - \mathbf{Q}\bar{\mathbf{l}} = \mathbf{0}.$$

The matrix

$$\mathbf{R} = \lambda\mathbf{P} + \mu\mathbf{Q}, \quad \lambda, \mu \in \mathbb{R}, \quad (\lambda, \mu) \neq (0, 0)$$

therefore satisfies

$$\mathbf{l}\mathbf{R} - \mathbf{R}\bar{\mathbf{l}} = \mathbf{0}\,.$$

Hence, if $\det \mathbf{R} > 0$, then $\mathbf{R}$ is a plane matrix determining a plane R of the *hyperbolic pencil* containing P and Q. Because of VII.3(8) with $\delta = \delta(P, Q)$ we have

$$\det \mathbf{R} = \lambda^2 + \mu^2 + \lambda\mu \operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}}) = \lambda^2 + \mu^2 + 2\,\lambda\mu \cosh \delta\,.$$

Consequently $\det \mathbf{R} > 0$ if and only if μ/λ satisfies (1). That the whole pencil is obtained for these values of μ/λ is seen as in the case of points. Application of VII.3(9) yields

$$(5) \qquad -\frac{\mu}{\lambda} = \frac{\sinh \delta(P, R)}{\sinh \delta(Q, R)}\,.$$

If $\det \mathbf{R} < 0$, so μ/λ satisfies (3), $\mathbf{r} = i\mathbf{R}$ is a point matrix determining a point of L. Again, all points of L are obtained for the values of μ/λ in question, and the analogue of (4) is valid.

Suppose now that the planes P and Q are parallel, and concordantly oriented. Let $\mathbf{P}$ and $\mathbf{Q}$ denote the normalized matrices determining them. Then

$$\mathbf{R} = \lambda\mathbf{P} + \mu\mathbf{Q}\,, \quad \lambda, \mu \in \mathbb{R}\,, \quad (\lambda, \mu) \neq (0, 0)\,,$$

is a regular plane matrix for all $\mu/\lambda \neq -1$ since

$$\det \mathbf{R} = \lambda^2 + \mu^2 + \lambda\mu \operatorname{tr}(\mathbf{Q}\bar{\mathbf{P}}) = \lambda^2 + \mu^2 + 2\,\lambda\mu\,.$$

$\mathbf{P} - \mathbf{Q}$ determines the improper plane coinciding with the common improper point of P and Q. The plane R determined by $\mathbf{R}$ belongs to the *parabolic pencil* containing P and Q, and all planes of this pencil are obtained. To see this assume P and Q to be the vertical e-half-planes bounded by the real axis and its parallel through i. Then

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix},$$

and the normalized matrix determining R is

$$\frac{1}{\lambda + \mu}\mathbf{R} = \begin{pmatrix} 1 & \dfrac{2\,\mu i}{\lambda + \mu} \\ 0 & 1 \end{pmatrix}.$$

Hence, R is the vertical e-half-plane bounded by the parallel to the real axis through $\mu i/(\lambda + \mu)$. Since $\mu/(\lambda + \mu)$ runs through the whole of $\mathbb{R}$, the statement follows. A geometrical interpretation of μ/λ can be obtained as follows. Choose a horosphere H with center at the common improper point of P and Q. In the sense

of the Euclidean metric induced on H the horocycles $P \cap H$, $Q \cap H$, $R \cap H$ are parallel lines. Denoting their distances in this metric provided with signs according to an orientation of a common normal, by $d(\ ,\ )$, we have

$$(6) \qquad -\frac{\mu}{\lambda} = \frac{d(P,R)}{d(Q,R)}.$$

With P and Q specialized as above and letting H be the horizontal e-plane through j, the distances $d(\ ,\ )$ equal the e-distances of the e-half-planes in question. Hence $d(P,R) = \mu/(\lambda+\mu)$, $d(Q,R) = -\lambda/(\lambda+\mu)$, so (6) holds.

Suppose now that the planes P and Q intersect in a line L. Choose orientations and let $\mathbf{P}, \mathbf{Q}, \mathbf{l}$ be normalized matrices determining P, Q, and L. By VII.2(8) we have

$$\mathbf{lP} + \mathbf{P\bar{l}} = \mathbf{0}, \qquad \mathbf{lQ} + \mathbf{Q\bar{l}} = \mathbf{0}.$$

Hence, $\mathbf{R} = \lambda\mathbf{P} + \mu\mathbf{Q}$, $\lambda, \mu \in \mathbb{R}$, $(\lambda,\mu) \neq (0,0)$, satisfies

$$\mathbf{lR} + \mathbf{R\bar{l}} = \mathbf{0}.$$

Since, by VII.3(14),

$$\det \mathbf{R} = \lambda^2 + \mu^2 + \lambda\mu \operatorname{tr}(\mathbf{Q\bar{P}}) = \lambda^2 + \mu^2 + 2\lambda\mu \cos\varphi(P,Q) > 0$$

for all $(\lambda,\mu) \neq (0,0)$, $\mathbf{R}$ is a plane matrix determining a plane of the *elliptic pencil* containing P and Q. By means of VII.3(15) one obtains

$$(7) \qquad -\frac{\mu}{\lambda} = \frac{\sin\varphi(P,R)}{\sin\varphi(Q,R)}.$$

To see that all planes of the pencil are obtained, one may write (7)

$$-\frac{\mu}{\lambda} = \sin\varphi(P,Q)\cot\varphi(Q,R) + \cos\varphi(P,Q)$$

and infer that, if $-\mu/\lambda$ runs through $\mathbb{R}$, then so does $\cot\varphi(Q,R)$.

As an immediate consequence of the preceding we note:

Three points are collinear if and only if point matrices determining them are linearly dependent over $\mathbb{R}$.

Three planes belong to the same pencil if and only if plane matrices determining them are linearly dependent over $\mathbb{R}$.

We notice that three regular or singular point (plane) matrices which are linearly dependent over $\mathbb{C}$ are actually linearly dependent over $\mathbb{R}$. To see this, suppose point matrices $\mathbf{p}, \mathbf{q}, \mathbf{r}$ satisfy

$$\lambda\mathbf{p} + \mu\mathbf{g} + \nu\mathbf{r} = \mathbf{0}$$

with complex numbers λ, μ, ν, which are not proportional to reals. Then we also have

$$\lambda \mathbf{p}^{\sim} + \mu \mathbf{g}^{\sim} + \nu \mathbf{r}^{\sim} = -\lambda \bar{\mathbf{p}} - \mu \bar{\mathbf{g}} - \nu \bar{\mathbf{r}} = \mathbf{0}\,,$$

and hence

$$(\lambda + \bar{\lambda})\mathbf{p} + (\mu + \bar{\mu})\mathbf{g} + (\nu + \bar{\nu})\mathbf{r} = 0\,.$$

In the same way the statement is seen to hold for plane matrices.

VII.5 Bundles of points and planes

Let p, q, r be non-collinear proper points and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ the normalized point matrices determining them. As mentioned at the end of the previous section, these matrices are linearly independent. We are going to prove that

(1) $$\mathbf{H} = \mathbf{r}\bar{\mathbf{q}}\mathbf{p} - \mathbf{p}\bar{\mathbf{q}}\mathbf{r}$$

is a plane matrix determining the plane H containing the given points.

The matrix $\mathbf{r}\bar{\mathbf{q}}\mathbf{p}j$ determines a reversal f^*, namely the product of the point reflections in p, q, r taken in this order. Since p, q, r are non-collinear, f^* is not a point reflection. Hence, according to VII.2,

$$\mathbf{r}\bar{\mathbf{q}}\mathbf{p} + \overline{(\mathbf{r}\bar{\mathbf{q}}\mathbf{p})^{\sim}} = \mathbf{r}\bar{\mathbf{q}}\mathbf{p} - \mathbf{p}\bar{\mathbf{q}}\mathbf{r}$$

is a plane matrix determining the mirror of f^*, and this is the plane H.

Using VII.2(4) we see that

(2) $$\operatorname{tr}(\mathbf{s}\bar{\mathbf{H}}) = 0$$

for every point matrix $\mathbf{s}$ determining a proper or improper point of H. In particular

$$\operatorname{tr}(\mathbf{p}\bar{\mathbf{H}}) = 0\,, \quad \operatorname{tr}(\mathbf{q}\bar{\mathbf{H}}) = 0\,, \quad \operatorname{tr}(\mathbf{r}\bar{\mathbf{H}}) = 0\,.$$

Consequently (2) is satisfied for all

(3) $$\mathbf{s} = \lambda\mathbf{p} + \mu\mathbf{q} + \nu\mathbf{r}\,, \quad \lambda, \mu, \nu \in \mathbb{R}\,, \quad (\lambda, \mu, \nu) \neq (0,0,0)\,,$$

and no other matrices $\mathbf{s}$ such that $\mathbf{s}^{\sim} = -\bar{\mathbf{s}}$. The latter statement follows from the fact that (2) may be considered as a linear homogeneous equation for the real unknowns $\operatorname{Re} s_{11}, \operatorname{Im} s_{11}, s_{12}, s_{21}$ which has three linearly independent solutions $\mathbf{p}, \mathbf{q}, \mathbf{r}$.

Now (3) is a point matrix if

$$\begin{aligned}\det \mathbf{s} &= \det(\lambda\mathbf{p} + \mu\mathbf{q} + \nu\mathbf{r}) \\ &= \lambda^2 + \mu^2 + \nu^2 - \lambda\mu \operatorname{tr}(\mathbf{p}\bar{\mathbf{q}}) - \mu\nu \operatorname{tr}(\mathbf{q}\bar{\mathbf{r}}) - \nu\lambda \operatorname{tr}(\mathbf{r}\bar{\mathbf{p}}) \\ &\geq 0.\end{aligned}$$

Hence, for the λ, μ, ν satisfying this inequality, (3) yields precisely all proper and improper points of H. According to VII.3(1) we have for instance

$$\operatorname{tr}(\mathbf{p}\bar{\mathbf{q}}) = -2\cosh\delta(\mathrm{p},\mathrm{q}) < -2\,.$$

This shows that the quadratic form in λ, μ, ν is indefinite since its value for $\lambda = 1$, $\mu = -1$, $\nu = 0$ is negative. If $\det \mathbf{s} < 0$, $\mathbf{S} = i\,\mathbf{s}$ is a plane matrix satisfying

$$\operatorname{tr}(\mathbf{S}\bar{\mathbf{H}}) = 0\,.$$

According to VII.2(9) it then determines a plane orthogonal to H, and by the argument above one sees that all planes of *the hyperbolic bundle* of such planes are obtained.

The matrix (1), normalized by a positive factor, provides the plane H with an orientation. We are going to determine it. We start by observing that $\mathbf{H}$ changes sign if two of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are interchanged. This is obvious for $\mathbf{p}$ and $\mathbf{r}$. To see it for $\mathbf{p}$ and $\mathbf{q}$ consider

$$\mathbf{r}\bar{\mathbf{q}}\mathbf{p} - \mathbf{p}\bar{\mathbf{q}}\mathbf{r} + \mathbf{r}\bar{\mathbf{p}}\mathbf{q} - \mathbf{q}\bar{\mathbf{p}}\mathbf{r} = \mathbf{r}(\bar{\mathbf{q}}\mathbf{p} + \bar{\mathbf{p}}\mathbf{q}) - (\mathbf{p}\bar{\mathbf{q}} + \mathbf{q}\bar{\mathbf{p}})\mathbf{r}\,.$$

This, indeed, equals $\mathbf{0}$ since

$$\begin{aligned}\bar{\mathbf{q}}\mathbf{p} + \bar{\mathbf{p}}\mathbf{q} &= \bar{\mathbf{q}}\mathbf{p} + (\bar{\mathbf{q}}\mathbf{p})^{\sim} = \operatorname{tr}(\bar{\mathbf{q}}\mathbf{p})\,\mathbf{1}\,,\\ \mathbf{p}\bar{\mathbf{q}} + \mathbf{q}\bar{\mathbf{p}} &= \mathbf{p}\bar{\mathbf{q}} + (\mathbf{p}\bar{\mathbf{q}})^{\sim} = \operatorname{tr}(\mathbf{p}\bar{\mathbf{q}})\,\mathbf{1} = \operatorname{tr}(\bar{\mathbf{q}}\mathbf{p})\,\mathbf{1}\,.\end{aligned}$$

Hence, $\mathbf{H}$ changes sign under odd permutations of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and remains unchanged under even ones. This shows that the orientation of H depends on the sense of the triangle pqr, the vertices taken in this cyclical order. It will turn out that this sense together with the positive direction of a normal to H form a left-handed skrew, thus, opposite the orientation of the space.

Let $\mathbf{H}'$ denote the normalized plane matrix obtained by dividing $\mathbf{H}$ by a positive number. To determine it, we use that, according to VII.2(3),

$$\mathbf{p} = \mathbf{l}\mathbf{H}' = \mathbf{H}'\bar{\mathbf{l}}\,, \quad \mathbf{q} = \mathbf{m}\mathbf{H}' = \mathbf{H}'\bar{\mathbf{m}}\,, \quad \mathbf{r} = \mathbf{n}\mathbf{H}' = \mathbf{H}'\bar{\mathbf{n}}\,,$$

where $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ denote the normalized line matrices determining the normals L, M, N to H through p, q, r, respectively, oriented in accordance with H. Since $\mathbf{H}'\bar{\mathbf{H}}' = \mathbf{1}$, we obtain

$$\begin{aligned}\mathbf{H} = \mathbf{r}\bar{\mathbf{q}}\mathbf{p} - \mathbf{p}\bar{\mathbf{q}}\mathbf{r} &= \mathbf{n}\mathbf{H}'\bar{\mathbf{H}}'\mathbf{m}\mathbf{l}\mathbf{H}' - \mathbf{l}\mathbf{H}'\bar{\mathbf{H}}'\mathbf{m}\mathbf{n}\mathbf{H}'\\ &= (\mathbf{nml} - \mathbf{lmn})\mathbf{H}' = (\mathbf{nml} + (\mathbf{nml})^{\sim})\mathbf{H}'\\ &= \operatorname{tr}(\mathbf{nml})\mathbf{H}'\,.\end{aligned}$$

We conclude that $\operatorname{tr}(\mathbf{nml}) > 0$. Now by definition

$$am(L, M, N) = -\frac{1}{2}\operatorname{tr}(\mathbf{nml})$$

and, apart from the sign, $am(L, M, N)$ equals the vertex amplitude $am_v(\mathrm{pqr}) > 0$ of the triangle pqr (cf. VI.5). Since $am(L, M, N) < 0$, the sense of the triangle pqr and the direction of the lines must form a left-handed skrew, as claimed. Finally we note

$$(4) \qquad \mathbf{H} = 2\,am_v(\mathrm{pqr})\mathbf{H}',$$

$$(5) \qquad \det \mathbf{H} = 4\,am_v^2(\mathrm{pqr}).$$

Let now three proper planes P, Q, R, not belonging to one pencil, be given, and let $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ be normalized matrices determining them. As observed at the end of VII.4, the latter are linearly independent. The reversal f^* determined by

$$\mathbf{f}^* j = \mathbf{R}\bar{\mathbf{Q}}\mathbf{P} j$$

is the product of the plane-reflections in P, Q, R. It is not a plane-reflection since the planes do not belong to a pencil. Here the matrix

$$(6) \qquad \mathbf{f}^* - \bar{\mathbf{f}}^{*\sim} = \mathbf{R}\bar{\mathbf{Q}}\mathbf{P} - \mathbf{P}\bar{\mathbf{Q}}\mathbf{R}$$

is significant. As in the case of three points, it is seen that (6) changes sign if two of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ are interchanged. To interprete (6) three cases have to be distinguished:

a) If the planes P, Q, R have a proper point h in common, f^* is a rotary reflection since it leaves this point invariant. As shown in VII.2, (6) is a point matrix $\mathbf{h}$ determining h. According to VII.2(4) we have

$$(7) \qquad \mathrm{tr}(\mathbf{h}\bar{\mathbf{S}}) = 0$$

for every plane matrix $\mathbf{S}$ determining a plane through h. In particular,

$$\mathrm{tr}(\mathbf{h}\bar{\mathbf{P}}) = 0, \quad \mathrm{tr}(\mathbf{h}\bar{\mathbf{Q}}) = 0, \quad \mathrm{tr}(\mathbf{h}\bar{\mathbf{R}}) = 0.$$

Consequently, (7) is valid for all

$$(8) \qquad \mathbf{S} = \lambda\mathbf{P} + \mu\mathbf{Q} + \nu\mathbf{R}, \quad \lambda, \mu, \nu \in \mathbb{R}, \quad (\lambda, \mu, \nu) \neq (0,0,0),$$

and no other matrices $\mathbf{S}$ satisfying $\mathbf{S}^\sim = \bar{\mathbf{S}}$. The latter statement follows as in the case of point matrices. Now

$$(9) \qquad \begin{aligned} \det \mathbf{S} &= \det(\lambda\mathbf{P} + \mu\mathbf{Q} + \nu\mathbf{P}) \\ &= \lambda^2 + \mu^2 + \nu^2 + \lambda\mu\,\mathrm{tr}(\mathbf{P}\bar{\mathbf{Q}}) + \mu\nu\,\mathrm{tr}(\mathbf{Q}\bar{\mathbf{R}}) + \nu\lambda\,\mathrm{tr}(\mathbf{R}\bar{\mathbf{P}}) \end{aligned}$$

is a positive definite quadratic form, as will be shown below. Hence, all matrices (8) are plane matrices, and they determine precisely the planes of the *elliptic bundle* with vertex h.

Let $\mathbf{h}'$ denote the normalized point matrix determining h. It is obtained by dividing $\mathbf{h}$ by a positive or negative number according as the element h_{21} of $\mathbf{h}$ is positive or negative. This sign depends on the orientations of P, Q, R. Let L, M, N

be the oriented normals through h to P, Q, R, respectively, and $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ the normalized line matrices determining them. According to VII.2(2) we then have

$$\mathbf{P} = \mathbf{h}'\bar{\mathbf{l}} = -\mathbf{l}\mathbf{h}', \quad \mathbf{Q} = -\mathbf{h}'\bar{\mathbf{m}} = -\mathbf{m}\mathbf{h}', \quad \mathbf{R} = -\mathbf{h}'\bar{\mathbf{n}} = -\mathbf{n}\mathbf{h}'.$$

Since $\mathbf{h}'\bar{\mathbf{h}}' = -\mathbf{1}$, these relations yield

$$\begin{aligned}\mathbf{h} &= \mathbf{R}\bar{\mathbf{Q}}\mathbf{P} - \mathbf{P}\bar{\mathbf{Q}}\mathbf{R} = -\mathbf{n}\mathbf{h}'\bar{\mathbf{h}}'\mathbf{m}\mathbf{l}\mathbf{h}' + \mathbf{l}\mathbf{h}'\bar{\mathbf{h}}'\mathbf{m}\mathbf{n}\mathbf{h}' \\ &= (\mathbf{n}\mathbf{m}\mathbf{l} - \mathbf{l}\mathbf{m}\mathbf{n})\mathbf{h}' = (\mathbf{n}\mathbf{m}\mathbf{l} + (\mathbf{n}\mathbf{m}\mathbf{l})^{\sim})\mathbf{h}' \\ &= \operatorname{tr}(\mathbf{n}\mathbf{m}\mathbf{l})\mathbf{h}'.\end{aligned}$$

Now

$$-\frac{1}{2}\operatorname{tr}(\mathbf{n}\mathbf{m}\mathbf{l}) = am(L, M, N),$$

so

$$\mathbf{h} = -2\,am(L, M, N)\mathbf{h}', \tag{10}$$

$$\det \mathbf{h} = 4\,am'(L, M, N). \tag{11}$$

According to VI.5, $am(L, M, N)$ is the vertex amplitude of the spherical triangle the vertices of which are the intersections of the positive half-lines of L, M, N issuing from h with the sphere of radius 1 and centre h. If the half-lines taken in this order form a right-handed frame, $am(L, M, N) < 0$ as VI.5(19) shows.

b) If P, Q, R have an improper point $h \in \mathbb{C}_\infty$ in common, the product of the reflections in these planes is a parallel reflection with fixed point h. As shown at the end of VII.2,

$$\mathbf{h} = \mathbf{R}\bar{\mathbf{Q}}\mathbf{P} - (\mathbf{R}\bar{\mathbf{Q}}\mathbf{P})^{\sim} = \mathbf{R}\bar{\mathbf{Q}}\mathbf{P} - \mathbf{P}\bar{\mathbf{Q}}\mathbf{R}$$

is a singular point matrix determining h.

According to VII.2(4), a plane S determined by the matrix $\mathbf{S}$ has h on its horizon if and only if

$$\operatorname{tr}(\mathbf{h}\bar{\mathbf{S}}) = 0. \tag{12}$$

In particular,

$$\operatorname{tr}(\mathbf{h}\bar{\mathbf{P}}) = 0, \quad \operatorname{tr}(\mathbf{h}\bar{\mathbf{Q}}) = 0, \quad \operatorname{tr}(\mathbf{h}\bar{\mathbf{R}}) = 0.$$

Hence, (12) is satisfied for all

$$\mathbf{S} = \lambda\mathbf{P} + \mu\mathbf{Q} + \nu\mathbf{R}, \quad \lambda, \mu, \nu \in \mathbb{R}, \quad (\lambda, \mu, \nu) \neq (0,0,0), \tag{13}$$

and by no other matrices $\mathbf{S}$ with $\mathbf{S}^{\sim} = \bar{\mathbf{S}}$, which may be seen as in the case of three points. Here the form (9) is positive semidefinite of rank 2, as will be shown below. With the triplets $(\lambda, \mu, \nu) \neq (0,0,0)$ for which $\det \mathbf{S} = 0$, (12) determines h. Consequently, (13) determines precisely all planes of the *parabolic bundle* with improper vertex h.

(c) Suppose finally that P, Q, R have a common normal plane H. Here

$$\mathbf{H} = i\,[\mathbf{R}\bar{\mathbf{Q}}\mathbf{P} - \overline{(\mathbf{R}\bar{\mathbf{Q}}\mathbf{P})^{\sim}}] = i(\mathbf{R}\bar{\mathbf{Q}}\mathbf{P} - \mathbf{P}\bar{\mathbf{Q}}\mathbf{R})$$

is a plane matrix determining H. Indeed, as shown at the end of VII.2, it determines the plane through the axis of the product f^* of the reflections in P, Q, R and orthogonal to the mirror of f^*. It is obvious that the axis of f^* lies in H, and that f^* maps H onto itself with orientation reversed.

A plane S determined by the plane matrix $\mathbf{S}$ is orthogonal to H if and only if

$$\text{(14)} \qquad \operatorname{tr}(\mathbf{H}\bar{\mathbf{S}}) = 0$$

(cf. VII.2(9)). In particular,

$$\operatorname{tr}(\mathbf{H}\bar{\mathbf{P}}) = 0\,, \quad \operatorname{tr}(\mathbf{H}\bar{\mathbf{Q}}) = 0\,, \quad \operatorname{tr}(\mathbf{H}\bar{\mathbf{R}}) = 0$$

and hence (14) is satisfied precisely by all matrices (13), which is seen as in the other cases. Now $\mathbf{S}$ is a (regular or singular) plane matrix if $\det \mathbf{S} \geq 0$. The matrices (13) satisfying this condition determine precisely the *hyperbolic bundle* of planes orthogonal to H. It will be shown that the quadratic form (9) is indefinite in the present case. For those $\mathbf{S}$ for which $\det \mathbf{S} < 0$ the matrices $\mathbf{s} = i\mathbf{S}$ are point matrices determining the points of H since

$$\operatorname{tr}(\mathbf{H}\bar{\mathbf{s}}) = -\operatorname{tr}(\mathbf{s}\bar{\mathbf{H}}) = 0$$

because of (14) (cf. VII.2(4)).

Let $\mathbf{H}'$ be one of the two normalized plane matrices determining H. To determine its relation to $\mathbf{H}$ we introduce the normalized line matrices

$$\mathbf{l} = \mathbf{H}'\bar{\mathbf{P}} = -\mathbf{P}\bar{\mathbf{H}}'\,, \quad \mathbf{m} = \mathbf{H}'\bar{\mathbf{Q}} = -\mathbf{Q}\bar{\mathbf{H}}'\,, \quad \mathbf{n} = \mathbf{H}'\bar{\mathbf{R}} = -\mathbf{R}\bar{\mathbf{H}}'$$

which, according to VII.2(10), determine the lines in which H intersects P, Q, R, respectively, with orientations depending on those of P, Q, R and that of H determined by $\mathbf{H}'$. Since $\mathbf{P}\bar{\mathbf{P}} = 1$, $\mathbf{Q}\bar{\mathbf{Q}} = 1$, $\mathbf{R}\bar{\mathbf{R}} = 1$, $\mathbf{H}'\bar{\mathbf{H}}' = 1$, we have

$$\mathbf{P} = -\mathbf{l}\bar{\mathbf{H}}'\,, \quad \bar{\mathbf{Q}} = \bar{\mathbf{H}}'\mathbf{m}\,, \quad \mathbf{R} = -\mathbf{n}\mathbf{H}'\,,$$

hence

$$\begin{aligned} -i\mathbf{H} &= \mathbf{R}\bar{\mathbf{Q}}\mathbf{P} - \mathbf{P}\bar{\mathbf{Q}}\mathbf{R} \\ &= (\mathbf{nml} - \mathbf{lmn})\mathbf{H}' = (\mathbf{nml} + (\mathbf{nml})^{\sim})\mathbf{H}' \\ &= \operatorname{tr}(\mathbf{nml})\mathbf{H}'\,. \end{aligned}$$

Since $\operatorname{tr}(\mathbf{nml}) = -2\,am(L, M, N)$, we obtain

$$\text{(15)} \qquad \mathbf{H} = -2\,i\,am(L, M, N)\,\mathbf{H}'\,,$$

$$\text{(16)} \qquad \det \mathbf{H} = -4\,am^2(L, M, N)\,.$$

As observed at the end of VI.5, $am(L, M, N)$ is purely imaginary if L, M, N lie in a plane, which is the case here. The sign of the real factor $-2i\,am(L, M, N)$ depends of course on the orientations of L, M, N.

It remains to prove the statements concerning the signature of the quadratic form

$$\lambda^2 + \mu^2 + \nu^2 + \lambda\mu\,\mathrm{tr}(\mathbf{P}\bar{\mathbf{Q}}) + \mu\nu\,\mathrm{tr}(\mathbf{Q}\bar{\mathbf{R}}) + \nu\lambda\,\mathrm{tr}(\mathbf{R}\bar{\mathbf{P}})\,.$$

With

$$\mathbf{a} = \mathbf{R}\bar{\mathbf{P}}\,, \quad \mathbf{b} = \mathbf{Q}\bar{\mathbf{R}}\,, \quad \mathbf{c} = \mathbf{P}\bar{\mathbf{Q}}$$

the assumptions for the validity of I.3(6) are satisfied. For the discriminant of the form we therefore have

$$\text{(17)} \qquad \begin{vmatrix} 1 & \frac{1}{2}\mathrm{tr}(\mathbf{R}\bar{\mathbf{P}}) & \frac{1}{2}\mathrm{tr}(\mathbf{Q}\bar{\mathbf{R}}) \\ \frac{1}{2}\mathrm{tr}(\mathbf{R}\bar{\mathbf{P}}) & 1 & \frac{1}{2}\mathrm{tr}(\mathbf{P}\bar{\mathbf{Q}}) \\ \frac{1}{2}\mathrm{tr}(\mathbf{Q}\bar{\mathbf{R}}) & \frac{1}{2}\mathrm{tr}(\mathbf{P}\bar{\mathbf{Q}}) & 1 \end{vmatrix} = \tfrac{1}{2} - \tfrac{1}{4}\mathrm{tr}(\mathbf{R}\bar{\mathbf{P}}\mathbf{Q}\bar{\mathbf{R}}\mathbf{P}\bar{\mathbf{Q}})\,.$$

In case a), $\mathbf{R}\bar{\mathbf{P}}\mathbf{Q}j$ determines a rotary reflection and the right-hand side is positive because of IV.3(8). Further, the second order principle minor

$$\begin{vmatrix} 1 & \frac{1}{2}\mathrm{tr}(\mathbf{R}\bar{\mathbf{P}}) \\ \frac{1}{2}\mathrm{tr}(\mathbf{R}\bar{\mathbf{P}}) & 1 \end{vmatrix} = 1 - \tfrac{1}{4}\mathrm{tr}^2(\mathbf{R}\bar{\mathbf{P}}) > 0$$

since

$$\tfrac{1}{4}\mathrm{tr}^2(\mathbf{R}\bar{\mathbf{P}}) = \cos^2\varphi(P, R)$$

where $\varphi(P, R)$ denotes the angle from P to R. Hence, the form is positive definite, as claimed.

In case b), $\mathbf{R}\bar{\mathbf{P}}\mathbf{Q}j$ determines a parallel reflection, and the right-hand side of (17) vanishes because of IV.3(7). Since at most one of the angles $\varphi(P, Q)$, $\varphi(Q, R)$, $\varphi(R, P)$ can be 0 or π because P, Q, R do not belong to a parabolic pencil, there is a positive principle minor of second order of the determinant (17). This shows that the quadratic form is positive semidefinite.

In case c), $\mathbf{R}\bar{\mathbf{P}}\mathbf{Q}j$ determines a glide reflection, and the righthand side of (17) is negative because of IV.3(6). Consequently the quadratic form is indefinite, as claimed.

VII.6 Tetrahedra

Non-coplanar (proper or improper) points p_1, p_2, p_3, p_4 are the vertices of a tetrahedron T. To keep trace of signs in the sequel, we assume that they, taken in this order, determine a right-handed skrew. The matrix $\mathbf{p}_i$ determining p_i is as-

sumed to be normalized if p_i is proper. Let P_i denote the (necessarily proper) plane of the face opposite p_i, or if convenient this face itself, oriented such that p_i lies in the positive half-space. The normalized matrix determining it is denoted by $\mathbf{P}_i$. By E_{ij}, $i \neq j$, we denote the line containing the edge $p_i p_j$ oriented from p_i towards p_j and by $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$ the normalized matrix determining it. Further let $N_{i,j}$, $i \neq j$, be the normal to P_i through p_j, oriented, if proper, in accordance with P_i, and $\mathbf{n}_{i,j}$ the matrix determining it, normalized if $N_{i,j}$ is proper. We note that VII.2(1) yields

$$(1) \qquad \mathbf{n}_{i,j} = \mathbf{P}_i \bar{\mathbf{p}}_j = \mathbf{p}_j \bar{\mathbf{P}}_i\,, \quad i \neq j\,.$$

We denote the length of the edge $p_i p_j$ by $\sigma_{ij} = \sigma_{ji} \geq 0$, admitting $i = j$ and defining $\sigma_{ii} = 0$. According to VII.3(1) we then have

$$(2) \qquad \cosh \sigma_{ij} = -\tfrac{1}{2}\operatorname{tr}(\mathbf{p}_j \bar{\mathbf{p}}_i) = -\tfrac{1}{2}\operatorname{tr}(\mathbf{p}_i \bar{\mathbf{p}}_j).$$

Let $\varphi_{ij} = \varphi_{ji} \in \,]\!]0, \pi[\![\,$, $i \neq j$, be the dihedral angle of the faces P_i and P_j. It is convenient to define $\varphi_{ii} = \pi$. Since this angle φ_{ij} is, plus or minus, the supplement of the angle between P_i and P_j as defined in VII.3, we obtain, using VII.3(14),

$$(3) \qquad \cos \varphi_{ij} = -\tfrac{1}{2}\operatorname{tr}(\mathbf{P}_j \bar{\mathbf{P}}_i) = -\tfrac{1}{2}\operatorname{tr}(\mathbf{P}_i \bar{\mathbf{P}}_j).$$

We introduce further the notation $\psi_{i,l} \in [\![0, \pi[\![$ for the interior angle at the vertex $\mathbf{p}_i$ of the face P_l. Specializing V.3(2), we obtain

$$(4) \qquad \cos \psi_{i,l} = \tfrac{1}{2}\operatorname{tr}(\mathbf{e}_{ij}\mathbf{e}_{ik})$$

where (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$.

In the relations to be derived several types of amplitudes occur. In what follows (i, j, k, l) denotes any even permutation of $(1, 2, 3, 4)$, and the vertices are assumed proper

1) *The side amplitude of a face P_l:*

$$(5) \qquad \begin{aligned} am_s(P_l) &= am(E_{ij}, E_{jk}, E_{ki}) = -\tfrac{1}{2}\operatorname{tr}(\mathbf{e}_{ki}\mathbf{e}_{jk}\mathbf{e}_{ij}) \\ &= i \sin \psi_{i,l} \sinh \sigma_{ij} \sin \psi_{j,l}\,. \end{aligned}$$

(cf. VI.5(12)).

2) *The vertex amplitude of a face P_l:*

$$(6) \qquad \begin{aligned} am_v(P_l) &= am(N_{l,i}, N_{l,j}, N_{l,k}) = -\tfrac{1}{2}\operatorname{tr}(\mathbf{n}_{l,k}\mathbf{n}_{l,j}\mathbf{n}_{l,i}) \\ &= \sinh \sigma_{ij} \sin \psi_{j,l} \sinh \sigma_{j,k} \end{aligned}$$

(cf. VI.5(12)).

3) *The amplitude of the edges issuing from a vertex* p_l:

$$(7)\qquad \begin{aligned} am_e(\mathrm{p}_l) = am(E_{il}, E_{jl}, E_{kl}) &= -\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{e}_{kl}\,\mathbf{e}_{jl}\,\mathbf{e}_{il}) \\ &= -\sin\psi_{l,i}\sin\varphi_{ij}\sinh\psi_{l,j} \end{aligned}$$

(cf. VI.5(19)).

4) *The amplitude of the normals to the faces through the vertex* p_l:

$$(8)\qquad \begin{aligned} am_n(\mathrm{p}_l) = am(N_{i,l}, N_{j,l}, N_{k,l}) &= -\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{n}_{k,l}\,\mathbf{n}_{j,l}\,\mathbf{n}_{i,l}) \\ &= -\sin\varphi_{ij}\sin\psi_{l,i}\sin\varphi_{ik} \end{aligned}$$

(cf. VI.5(18)).

We note several relations between the matrices and quantities introced. In what follows it is assumed about the indices that (i,j,k,l) is an even permutation of $(1,2,3,4)$.

According to VII.3(3) we have

$$(9)\qquad 2i\sinh\sigma_{ij}\mathbf{e}_{ij} = \mathbf{p}_j\bar{\mathbf{p}}_i - \mathbf{p}_i\bar{\mathbf{p}}_j\,,$$

provided p_i, p_j are proper, and VII.3(16) yields

$$(10)\qquad 2\sin\varphi_{kl}\mathbf{e}_{ij} = \mathbf{P}_k\bar{\mathbf{P}}_l - \mathbf{P}_l\bar{\mathbf{P}}_k\,.$$

Here it has to be observed that the angle $\varphi(P_k, P_l)$ occurring there equals $-(\pi-\varphi_{kl})$.

By VII.5(1) and (4), observing the statement concerning the orientation of the plane involved, we obtain

$$(11)\qquad \begin{aligned} \mathbf{p}_k\bar{\mathbf{p}}_j\mathbf{p}_i - \mathbf{p}_i\bar{\mathbf{p}}_j\mathbf{p}_k &= -2\,am(N_{l,i}, N_{l,j}, N_{l,k}) \\ &= -2\,am_v(P_l)\mathbf{P}_l\,, \end{aligned}$$

provided p_i, p_j, p_k are proper. VII.3(6) and (10) yield

$$(12)\qquad \begin{aligned} \mathbf{P}_k\bar{\mathbf{P}}_j\mathbf{P}_i - \mathbf{P}_i\bar{\mathbf{P}}_j\mathbf{P}_k &= -2\,am(N_{i,l}, N_{j,l}, N_{k,l}) \\ &= -2\,am_n(\mathrm{p}_l)\mathbf{p}_l\,, \end{aligned}$$

provided p_l is proper.

Further we have use for the positive length a_l of the altitude of the tetrahedron from a proper vertex p_l. By VII.3(5) we have

$$(13)\qquad \sinh a_l = \frac{i}{2}\,\mathrm{tr}\,(\mathbf{P}_l\bar{\mathbf{p}}_l) = \frac{i}{2}\,\mathrm{tr}\,(\mathbf{p}_l\bar{\mathbf{P}}_l)$$

since p_l lies in the positive half-space of P_l. An expression for $\sinh a_l$ in terms of the quantities introduced can be obtaihed as follows. The altitude of length $a_{l,j,k}$

from p_l of the triangle $p_j p_k p_l$ is the hypotenuse of a right-angled triangle. Another side of which is a_l and the angle opposite to it is φ_{il}. Hence

$$\sinh a_l = \sin \varphi_{il} \sinh a_{l,jk} \,.$$

According to VI.5(11),

$$\begin{aligned} am_v(P_i) &= am_v(p_j p_k p_l) = \sinh \sigma_{jk} \sinh a_{l,jk} \,, \\ am_s(P_i) &= am_s(p_j p_k p_l) = i \sinh \psi_{l,i} \sinh a_{l,jk} \,, \end{aligned}$$

and consequently

$$\sinh a_l = am_v(P_i) \frac{\sin \varphi_{il}}{\sinh \sigma_{jk}} = -\, i\, am_s(P_i) \frac{\sin \varphi_{il}}{\sin \psi_{l,i}} \,. \tag{14}$$

With a tetrahedron T we associate two real numbers which will also be called amplitudes. The *vertex amplitude* of T is defined by

$$\begin{aligned} Am_v(T) &= \frac{i}{4} \operatorname{tr}(\mathbf{p}_4 \bar{\mathbf{p}}_3 \mathbf{p}_2 \bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_4 \mathbf{p}_3 \bar{\mathbf{p}}_2 \mathbf{p}_1) \\ &= -\frac{1}{2} \operatorname{Im} \operatorname{tr}(\mathbf{p}_4 \bar{\mathbf{p}}_3 \mathbf{p}_2 \bar{\mathbf{p}}_1) \,, \end{aligned} \tag{15}$$

provided all vertices are proper, and the *face amplitude* by

$$\begin{aligned} Am_f(T) &= \frac{i}{4} \operatorname{tr}(\mathbf{P}_4 \bar{\mathbf{P}}_3 \mathbf{P}_2 \bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_4 \mathbf{P}_3 \bar{\mathbf{P}}_2 \mathbf{P}_1) \\ &= -\frac{1}{2} \operatorname{Im} \operatorname{tr}(\mathbf{P}_4 \bar{\mathbf{P}}_3 \mathbf{P}_2 \bar{\mathbf{P}}_1) \,. \end{aligned} \tag{16}$$

We are going to show that the right-hand sides of (15) and (16) change signs under odd permutations, and thus remain unchanged under even permutations, of the subscripts $1, 2, 3, 4$. Obviously this holds for cyclic permutations since

$$\operatorname{tr}(\mathbf{p}_1 \bar{\mathbf{p}}_4 \mathbf{p}_3 \bar{\mathbf{p}}_2) = \operatorname{tr}(\bar{\mathbf{p}}_4 \mathbf{p}_3 \bar{\mathbf{p}}_2 \mathbf{p}_1) \,.$$

Further, using I.3(2),

$$\begin{aligned} \operatorname{tr}(\mathbf{p}_4 \bar{\mathbf{p}}_3 \mathbf{p}_1 \bar{\mathbf{p}}_2) - \operatorname{tr}(\bar{\mathbf{p}}_4 \mathbf{p}_3 \bar{\mathbf{p}}_1 \mathbf{p}_2) = {} & \operatorname{tr}(\mathbf{p}_4 \bar{\mathbf{p}}_3) \operatorname{tr}(\mathbf{p}_1 \bar{\mathbf{p}}_2) - \operatorname{tr}(\mathbf{p}_4 \bar{\mathbf{p}}_3 \mathbf{p}_2 \bar{\mathbf{p}}_1) \\ & - \operatorname{tr}(\bar{\mathbf{p}}_4 \mathbf{p}_3) \operatorname{tr}(\bar{\mathbf{p}}_1 \mathbf{p}_2) + \operatorname{tr}(\bar{\mathbf{p}}_4 \mathbf{p}_3 \bar{\mathbf{p}}_2 \mathbf{p}_1) \,, \end{aligned}$$

and this is the statement for the cycle $(1, 2)$ since $\operatorname{tr}(\mathbf{p}_1 \bar{\mathbf{p}}_2)$ and $\operatorname{tr}(\mathbf{p}_4 \bar{\mathbf{p}}_3)$ are real. The transposition $(1, 2)$ and the powers of the cycle $(1, 2, 3, 4)$ generate the symmetric group S_4.*) Indeed, transforming $(1, 2)$ by the powers of $(1, 2, 3, 4)$ one

*) Here, the terminology and the notation customary in the study of permutations is used.

obtains (2, 3), (3.4), (4, 1), and

$$(1,2)\,(2,3)\,(1,2) = (1,3)\,, \quad (2,3)\,(3,4)\,(2,3) = (2,4)\,.$$

Hence, all transpositions (i,j) are obtained. This proves the statement for the right-hand side of (15), and the same proof works for that of (16).

In the sequel several expressions for the amplitudes of a tetrahedron T will be derived. As before, (i, j, k, l) is assumed to be an even permutation of $(1, 2, 3, 4)$. In all relations involving $Am_v(T)$ the vertices of T have to be proper. For the other relations it will be indicated to what extent improper vertices are admissible.

Multiplying (11) by $\bar{\mathbf{p}}_l$ and (12) by $\bar{\mathbf{P}}_l$ from the right and taking traces, we obtain

$$\begin{aligned} \operatorname{tr}(\mathbf{p}_k\bar{\mathbf{p}}_j\mathbf{p}_i\bar{\mathbf{p}}_l) - \operatorname{tr}(\mathbf{p}_i\bar{\mathbf{p}}_j\mathbf{p}_k\bar{\mathbf{p}}_l) &= \operatorname{tr}(\bar{\mathbf{p}}_l\mathbf{p}_k\bar{\mathbf{p}}_j\mathbf{p}_i) - \operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{p}_j\bar{\mathbf{p}}_i) \\ &= -2\, am_v(P_l)\operatorname{tr}(\mathbf{P}_l\bar{\mathbf{p}}_l)\,, \end{aligned}$$

and, if p_l is proper,

$$\begin{aligned} \operatorname{tr}(\mathbf{P}_k\bar{\mathbf{P}}_j\mathbf{P}_i\bar{\mathbf{P}}_l) - \operatorname{tr}(\mathbf{P}_i\bar{\mathbf{P}}_j\mathbf{P}_k\bar{\mathbf{P}}_l) &= \operatorname{tr}(\bar{\mathbf{P}}_l\mathbf{P}_k\bar{\mathbf{P}}_j\mathbf{P}_i) - \operatorname{tr}(\mathbf{P}_l\bar{\mathbf{P}}_k\mathbf{P}_j\bar{\mathbf{P}}_i) \\ &= -2\, am_n(\mathrm{p}_l)\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{P}}_l)\,. \end{aligned}$$

According to (15), (16) and (13) we therefore have

(17) $$Am_v(T) = am_v(P_l)\sinh a_l\,,$$

(18) $$Am_f(T) = -\, am_n(\mathrm{p}_l)\sinh a_l\,.$$

The latter relation holds whenever p_l is proper. We note that

$$Am_v(T) > 0\,, \qquad Am_f(T) > 0$$

because of VI.5(11) and (20).

(17) and (18) yield an analogue of the Law of Sines:

(19) $$\frac{Am_v(T)}{Am_f(T)} = -\frac{am_v(P_l)}{am_n(\mathrm{p}_l)}\,, \quad l = 1, 2, 3, 4\,.$$

A different form of it may be obtained as follows. Applying (6) and (8), we obtain

$$-\frac{am_v(P_l)}{am_n(\mathrm{p}_l)} = \frac{\sinh\sigma_{ij}\sin\psi_{j,l}\sinh\sigma_{jk}}{\sin\varphi_{ij}\sin\psi_{l,i}\sin\varphi_{ik}}\,.$$

Now the Law of Sines applied to the triangle $\mathrm{p}_j\mathrm{p}_k\mathrm{p}_l$ yields

$$\frac{\sinh\sigma_{jk}}{\sin\psi_{l,i}} = \frac{\sinh\sigma_{kl}}{\sin\psi_{j,i}}\,,$$

and application to the spherical triangle determined by the edges through p_j yields

$$\frac{\sin\psi_{j,l}}{\sin\varphi_{ik}} = \frac{\sin\psi_{j,i}}{\sin\varphi_{kl}}.$$

We therefore obtain

$$\frac{Am_v(T)}{Am_f(T)} = \frac{\sinh\sigma_{ij}\sinh\sigma_{kl}}{\sin\varphi_{ij}\sin\varphi_{kl}} \tag{20}$$

for any even permutation (i,j,k,l) of $(1,2,3,4)$.

We return to (17) and (18). Application of (14) yields

$$Am_v(T) = am_v(P_l)\,am_v(P_i)\,\frac{\sin\varphi_{il}}{\sinh\sigma_{jk}},$$

$$Am_f(T) = i\,am_n(\mathrm{p}_l)\,am_s(P_i)\,\frac{\sin\varphi_{il}}{\sin\psi_{l,i}}.$$

Using (6) as it stands and with the permutation $(i,j,k,l)\mapsto(j,l,k,i)$ of the subscripts, we obtain

$$Am_v(T) = \sinh\sigma_{ij}\sin\psi_{j,l}\sinh\sigma_{jl}\sin\psi_{j,i}\sinh\sigma_{jk}\sin\varphi_{il}$$

and by (7) with permutation $(i,j,k,l)\mapsto(k,l,i,j)$

$$Am_v(T) = -\sinh\sigma_{ij}\sinh\sigma_{jk}\sinh\sigma_{jl}\,am_e(\mathrm{p}_j). \tag{21}$$

Application of (8) yields

$$Am_f(T) = -i\sin\varphi_{ij}\sin\varphi_{ik}\sin\varphi_{il}\,am_s(P_i). \tag{22}$$

Further expressions for the amplitudes of T involve the distance and the angle between opposite side lines E_{ij} and E_{kl}. Let M be the common normal of E_{ij} and E_{kl} oriented from E_{ij} towards E_{kl}. We consider the width of the double cross $(E_{ij}, E_{kl}; M)$. We denote it by $\delta_{ij,kl} - \chi_{ij,kl}\,i$, $\delta_{ij,kl} > 0$ being the distance and $-\chi_{ij,kl}$ with $\chi_{ij,kl}\in\,]0,\pi[$ the angle under the orientation choosen. According to V.3(2) we have

$$\cosh(\delta_{ij,kl} - \chi_{ij,kl}\,i) = -\tfrac{1}{2}\operatorname{tr}(\mathbf{e}_{kl}\mathbf{e}_{ij})$$

and hence

$$\sinh\delta_{ij,kl}\sin\chi_{ij,kl} = \tfrac{1}{2}\operatorname{Im}\operatorname{tr}(\mathbf{e}_{kl}\mathbf{e}_{ij}). \tag{23}$$

The left-hand side is frequently called the *moment* of the two lines. Let it be denoted by *mom* (E_{ij}, E_{kl}).

We are going to prove that

$$Am_v(T) = \sinh\sigma_{ij}\sinh\sigma_{kl}\,mom(E_{ij}, E_{kl}). \tag{24}$$

Considering that VII.3(2) implies

$$\text{(26)}\qquad \sinh\sigma_{ij} = \frac{i}{2}\operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij}), \quad \sinh\sigma_{kl} = \frac{i}{2}\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{kl}),$$

we start with the following expression, using VII.2(5) several times:

$$\begin{aligned}
&\operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij})\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{kl})\operatorname{tr}(\mathbf{e}_{kl}\mathbf{e}_{ij})\\
&\qquad = \operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij})\,[-\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{ij}) + \operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{kl}\mathbf{e}_{ij}\mathbf{e}_{kl})]\\
&\qquad = \operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij})\,[-\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{ij}) + \operatorname{tr}(\mathbf{e}_{kl}\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{ij}\mathbf{e}_{kl})]\\
&\qquad = -2\operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij})\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{ij})\\
&\qquad = -2\,[\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{ij}\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij}) + \operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{p}_i\bar{\mathbf{p}}_j)]\\
&\qquad = 2\,[\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{p}_j\bar{\mathbf{p}}_i) - \operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{p}_i\bar{\mathbf{p}}_j)]\,.
\end{aligned}$$

Since $\operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij})$ and $\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{kl})$ are purely imaginary, the complex conjugate of the expression equals

$$\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{ij})\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{kl})\overline{\operatorname{tr}(\mathbf{e}_{kl}\mathbf{e}_{ij})} = 2\,[\operatorname{tr}(\bar{\mathbf{p}}_l\mathbf{p}_k\bar{\mathbf{p}}_j\mathbf{p}_i) - \operatorname{tr}(\bar{\mathbf{p}}_l\mathbf{p}_k\bar{\mathbf{p}}_i\mathbf{p}_j)]\,.$$

Subtracting this from the expression above and observing that the right-hand side of (15) is alternating we obtain

$$\begin{aligned}
&\operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i\mathbf{e}_{ij})\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{e}_{kl})\,[\operatorname{tr}(\mathbf{e}_{kl}\mathbf{e}_{ij}) - \overline{\operatorname{tr}(\mathbf{e}_{kl}\mathbf{e}_{ij})}]\\
&\qquad = 4\,[\operatorname{tr}(\mathbf{p}_l\bar{\mathbf{p}}_k\mathbf{p}_j\bar{\mathbf{p}}_i) - \operatorname{tr}(\bar{\mathbf{p}}_l\mathbf{p}_k\bar{\mathbf{p}}_j\mathbf{p}_i)] = -16\,i\,Am_v(T)\,,
\end{aligned}$$

and this is the statement (24) because of (23) and (26).

By means of (20) one obtains from (24)

$$\text{(27)}\qquad Am_f(T) = \sin\varphi_{ij}\sin\varphi_{kl}\,mom(E_{ij}, E_{kl})\,.$$

Finally we derive expressions for $Am_v(T)$ in terms of the edge lengths σ_{ij} and for $Am_f(T)$ in terms of the dihedral angles φ_{ij}.

Since $\frac{i}{4}\operatorname{tr}(\mathbf{p}_4\bar{\mathbf{p}}_3\mathbf{p}_2\bar{\mathbf{p}}_1 - \bar{\mathbf{p}}_4\mathbf{p}_3\bar{\mathbf{p}}_2\mathbf{p}_1)$ clearly is a multi-linear and, as shown above, an alternating function of the real vectors

$$(\operatorname{Re}p_{i,11}, p_{i,12}, p_{i,21}, \operatorname{Im}p_{i,11})\,,$$

it must be proportional to the determinant with these vectors as rows, so

$$\begin{aligned}
Am_v(T) &= \lambda\det(\operatorname{Re}p_{i,11}, p_{i,12}, p_{i,21}, \operatorname{Im}p_{i,11})_{i=1,\dots,4}\\
&= -\frac{\lambda i}{2}\det(p_{i,11}, p_{i,12}, p_{i,21}, \bar{p}_{i,11})_{i=1,\dots,4}
\end{aligned}$$

To determine the factor λ we may insert any four linearly independent point

matrices, for instance

$$\mathbf{p}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},$$

$$\mathbf{p}_4 = \begin{pmatrix} i & -1 \\ 2 & i \end{pmatrix}.$$

For these one gets $Am_v(T) = \frac{3}{4}$, and the first determinant above equals $-\frac{3}{2}$, so $\lambda = -\frac{1}{2}$ and consequently

$$Am_v(T) = \frac{i}{4}\det(p_{i,11}, p_{i,12}, p_{i,21}, \bar{p}_{i,11})_{i=1,\dots,4}\,.$$

The determinant on the right-hand side equals

$$\det(\bar{p}_{i,11}, p_{i,21}, p_{i,12}, p_{i,11})_{i=1,\dots,4}\,.$$

Multiplying the two determinants rows by rows one obtains

$$Am_v^2(T) = -\tfrac{1}{16}\det(\operatorname{tr}(\mathbf{p}_j\bar{\mathbf{p}}_i))_{i,j=1,\dots,4}$$

since

$$\operatorname{tr}\left[\begin{pmatrix} p_{j,11} & p_{j,12} \\ p_{j,21} & -\bar{p}_{j,11} \end{pmatrix}\begin{pmatrix} \bar{p}_{i,11} & p_{i,12} \\ p_{i,21} & -p_{i,11} \end{pmatrix}\right]$$
$$= p_{j,11}\bar{p}_{i,11} + p_{j,12}p_{i,21} + p_{j,21}p_{i,12} + \bar{p}_{j,11}p_{i,11}\,.$$

Now by (2)

(28) $$Am_v^2(T) = -\det(\cosh\sigma_{ij})_{i,j=1,\dots,4}\,.$$

Analogously $Am_f(T)$ can be dealt with. One obtains

$$Am_f(T) = \frac{i}{4}\det(P_{i,11}, P_{i,12}, P_{i,21}, \bar{P}_{i,11})_{i=1,\dots,4}$$

and

$$Am_f^2(T) = -\tfrac{1}{16}\det(\operatorname{tr}(\mathbf{P}_j\bar{\mathbf{P}}_i))_{i,j=1,\dots,4}\,.$$

Hence by (3)

(29) $$Am_f^2(T) = -\det(\cos\varphi_{ij})_{i,j=1,\dots,4}\,.$$

Here the vertices need not be proper.

The results obtained permit to answer the following question: What are the conditions six positive numbers $\sigma_{ij} = \sigma_{ji}$, $i,j = 1,2,3,4$, $i \neq j$, have to satisfy in order that they are the edge lengths of a tetrahedron?

Necessary conditions are obviously that the numbers which are to be the sides

of a face satisfy the triangle inequalities

$$(30) \qquad \sigma_{ij} < \sigma_{ik} + \sigma_{jk} ,$$

where i, j, k are any three distinct ones of the numbers 1, 2, 3, 4. Further, because of (28),

$$(31) \qquad \det(\cosh \sigma_{ij})_{i,j=1,\ldots,4} < 0 .$$

We are going to show that these conditions are also sufficient.

Because of (30) the triangles which are to be the faces of a tetrahedron exist. Consider three of them which are required to have a vertex, p_1 say, in common. Their sides have the lengths $(\sigma_{13}, \sigma_{34}, \sigma_{41})$, $(\sigma_{12}, \sigma_{24}, \sigma_{41})$, $(\sigma_{12}, \sigma_{23}, \sigma_{31})$. Let $\psi_{1,2}$, $\psi_{1,3}$, $\psi_{1,4}$ denote their interior angles at the vertex to be common. The triangles can be joined to form a trihedron, and thus the tetrahedron required, provided there exists a spherical triangle with these angles as sides, that is, if their sum is less than 2π and they satisfy the triangle inequalities. It has to be shown that these conditions are a consequence of (31).

The Law of Cosines applied to the triangles yields

$$\cosh \sigma_{jk} = \cosh \sigma_{1j} \cosh \sigma_{1k} - \sinh \sigma_{1j} \sinh \sigma_{1k} \cos \psi_{1,l} ,$$

where (j, k, l) is an even permutation of (2, 3, 4). Multiplying the first row of the determinant in (31) by $\cosh \sigma_{12}$, by $\cosh \sigma_{13}$, by $\cos \sigma_{14}$, and subtracting it from the second, third, and fourth row, respectively, and using the preceding equation, we obtain

$$\det(\cosh \sigma_{ij})_{i,j=1,\ldots,4} =$$

$$= \begin{vmatrix} -\sinh^2 \sigma_{12} & \cosh \sigma_{23} - \cosh \sigma_{12} \cosh \sigma_{13} & \cosh \sigma_{24} - \cosh \sigma_{12} \cosh \sigma_{14} \\ \cosh \sigma_{23} - \cosh \sigma_{12} \cosh \sigma_{13} & -\sinh^2 \sigma_{13} & \cosh \sigma_{34} - \cosh \sigma_{13} \cosh \sigma_{14} \\ \cosh \sigma_{24} - \cosh \sigma_{12} \cosh \sigma_{14} & \cosh \sigma_{34} - \cosh \sigma_{13} \cosh \sigma_{14} & -\sinh^2 \sigma_{14} \end{vmatrix}$$

$$= - \begin{vmatrix} \sinh^2 \sigma_{12} & \sinh \sigma_{12} \sinh \sigma_{13} \cos \psi_{1,4} & \sinh \sigma_{12} \sinh \sigma_{14} \cos \psi_{1,3} \\ \sinh \sigma_{12} \sinh \sigma_{13} \cos \psi_{1,4} & \sinh^2 \sigma_{13} & \sinh \sigma_{13} \sinh \sigma_{14} \cos \psi_{1,2} \\ \sinh \sigma_{12} \sinh \sigma_{14} \cos \psi_{1,3} & \sinh \sigma_{13} \sinh \sigma_{14} \cos \psi_{1,2} & \sinh^2 \sigma_{14} \end{vmatrix}$$

$$= -\sinh^2 \sigma_{12} \sinh^2 \sigma_{13} \sinh^2 \sigma_{14} \begin{vmatrix} 1 & \cos \psi_{1,4} & \cos \psi_{1,3} \\ \cos \psi_{1,4} & 1 & \cos \psi_{1,2} \\ \cos \psi_{1,3} & \cos \psi_{1,2} & 1 \end{vmatrix}$$

Consequently, the condition (31) says that the last determinant has to be positive. Considering VI.5(21) we see that it amounts to

$$\sin\tfrac{1}{2}(\psi_{1,2}+\psi_{1,3}+\psi_{1,4})\sin\tfrac{1}{2}(-\psi_{1,2}+\psi_{1,3}+\psi_{1,4})\cdot$$
$$\cdot\sin\tfrac{1}{2}(\psi_{1,2}-\psi_{1,3}+\psi_{1,4})\sin\tfrac{1}{2}(\psi_{1,2}+\psi_{1,3}-\psi_{1,4})>0\,.$$

Suppose $\psi_{1,2}+\psi_{1,3}+\psi_{1,4}>2\pi$, so the first factor were negative. Then

$$2\pi>-\psi_{1,2}+\psi_{1,3}+\psi_{1,4}=\psi_{1,2}+\psi_{1,3}+\psi_{1,4}-2\psi_{1,2}>0$$

and the analogues. Hence, the three last factors would be positive. It follows that the condition implies that the first factor is positive, so the sum of the angles is less than 2π as claimed. To prove the triangle inequalities we have to show that the three last factors are positive since we have

$$-\psi_{1,2}+\psi_{1,3}+\psi_{1,4}<2\pi$$

and the analogues. Now

$$\tfrac{1}{2}(-\psi_{1,2}+\psi_{1,3}+\psi_{1,4})+\tfrac{1}{2}(\psi_{1,2}-\psi_{1,3}+\psi_{1,4})=\psi_{1,4}>0$$

and the analogues, and this implies that at most one of the factors can be negative. Hence all of them must be positive, so the triangle inequalities hold. This finishes the proof of the statement.

We mention another formulation of it:

Six positive numbers σ_{ij}, $i,j=1,2,3,4$, $i\neq j$, are the edge lengths of a tetrahedron if and only if the quadratic form

$$\sum_{i,j=1}^{4} x_i x_j \cosh\sigma_{ij}\,,$$

where $\sigma_{ii}=0$, has signature -2.

To show this, we have to determine the signs of the principal minors of the discriminant $\det(\cosh\sigma_{ij})$. The principal minors of order 1 are positive, those of order 2 clearly negative. A principal minor of order 3, for instance,

$$\begin{vmatrix} 1 & \cosh\sigma_{12} & \cosh\sigma_{13} \\ \cosh\sigma_{12} & 1 & \cosh\sigma_{23} \\ \cosh\sigma_{13} & \cosh\sigma_{23} & 1 \end{vmatrix} =$$

$$=4\sinh\tfrac{1}{2}(\sigma_{12}+\sigma_{13}+\sigma_{23})\sinh\tfrac{1}{2}(-\sigma_{12}+\sigma_{13}+\sigma_{23})$$
$$\cdot\sinh\tfrac{1}{2}(\sigma_{12}-\sigma_{13}+\sigma_{23})\sinh\tfrac{1}{2}(\sigma_{12}+\sigma_{13}-\sigma_{23})$$

is positive if and only if the triangle inequalities (30) are satisfied. Since the latter and (31) are necessary and sufficient for the existence of a tetrahedron as required, so are: the signs of the principal minors of orders 1, 2, 3, 4 are $+$, $-$, $+$, $-$. This proves the statement.

It is natural to pose the "dual" problem: What are the conditions six numbers $\varphi_{ij} = \varphi_{ji} \in]0, \pi[$, $i, j = 1, 2, 3, 4$, $i \neq j$ have to satisfy in order that they are the dihedral angles of a tetrahedron?
Since φ_{ij}, φ_{jk}, φ_{ki}, where i, j, k are any three distinct ones of the numbers 1, 2, 3, 4 have to be the angles of a spherical triangle, necessary conditions are

$$\varphi_{ij} + \varphi_{jk} + \varphi_{ki} > \pi .$$

Another one,

$$\det (\cos \varphi_{ij})_{i,j=1,\ldots,4} < 0$$

is a consequence of (29). However, it turns out that these conditions are not sufficient. The missing one seems to be very involved.

Notes to Chapter VII

The reader will find the treatment of points and planes in this chapter rather complicated. Formally it is certainly much simpler to use the projective model. However the geometrical interpretation of a formula in this model depends on the sign of a quantity involved or whether it is real or imaginary. The method used here does not have these disadvantages. It allows to express simply distances and angles of points, lines and planes.

The amplitudes of a tetrahedron have been introduced by Coolidge [3] 178–180. He proved also most of the relations dealt with in Section 6.

VIII. Spherical Surfaces

VIII.1 Equations of spherical surfaces

An equation of a spherical surface S may be written

$$(1) \qquad \alpha(x\bar{x}+\xi^2)-\bar{b}x-b\bar{x}-2\beta\xi+\gamma=0\,,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $b \in \mathbb{C}$, $(\alpha, b, \beta) \neq (0,0,0)$. These coefficients are determined up to a non-zero real factor. It will often be convenient to assume them chosen such that $\alpha \geq 0$.

If $\alpha = 0$, (1) determines an e-plane with normal vector $b+\beta j$. If $\alpha \neq 0$, (1) may be written

$$\left(x+\xi j-\frac{b+\beta j}{\alpha}\right)\left(\bar{x}-\xi j-\frac{\bar{b}-\beta j}{\alpha}\right)=\frac{b\bar{b}+\beta^2-\alpha\gamma}{\alpha^2}$$

and determines a (possibly degenerate) e-sphere with e-center $(b+\beta j)/\alpha$ and e-radius

$$(2) \qquad r=\left(\frac{b\bar{b}+\beta^2-\alpha\gamma}{\alpha^2}\right)^{\frac{1}{2}},$$

provided

$$(3) \qquad b\bar{b}+\beta^2-\alpha\gamma \geq 0\,.$$

This inequality is always assumed to be satisfied.

Equation (1) determines a *sphere* (in particular a proper point, a *point-sphere*) if and only if $\beta/\alpha > 0$ and $r^2 < \beta^2/\alpha^2$ equivalently, if and only if

$$\alpha\beta > 0\,, \qquad \alpha\gamma - b\bar{b} > 0\,.$$

The *centre* of the sphere is

$$(4) \qquad \frac{b}{\alpha}+\sqrt{\frac{\beta^2}{\alpha^2}-r^2}\; j=\frac{b}{\alpha}+\sqrt{\frac{\alpha\gamma-b\bar{b}}{\alpha^2}}\; j.$$

(cf. IV.5(1), the c and γ_e there are here b/α and β/α, respectively) and for its *radius* ϱ we have, assuming $\alpha > 0$ and thus $\beta > 0$,

$$(5) \qquad \tanh\varrho=\frac{\alpha r}{\beta}=\sqrt{\frac{b\bar{b}+\beta^2-\alpha\gamma}{\beta^2}}\,, \qquad \cosh\varrho=\frac{\beta}{\sqrt{\alpha\gamma-b\bar{b}}}\,.$$

If $r = 0$, (1) determines the point $(b+\beta j)/\alpha$.

The equation (1) determines a *horosphere* if and only if $0 < r = \beta/\alpha$ or $\alpha = 0$, $b = 0$, $\gamma/\beta > 0$, equivalently,

$$\alpha\gamma - b\bar{b} = 0 \quad \text{and} \quad \beta/\alpha > 0 \quad \text{if} \quad \alpha \neq 0, \qquad \gamma/\beta > 0 \quad \text{if} \quad \alpha = 0. \tag{6}$$

The equation (1) determines an *equidistant surface* if and only if $\alpha \neq 0$, $\beta \neq 0$, $r^2 > \beta^2/\alpha^2$ or $\alpha = 0$, $b \neq 0$, $\beta \neq 0$, hence, if and only if

$$\alpha\gamma - b\bar{b} < 0, \qquad \beta \neq 0.$$

An equation of the axial plane is obtained by setting $\beta = 0$ in (1).

Let $\delta \neq 0$ and $\varphi \in \left]-\pi/2, \pi/2\right[\setminus \{0\}$ denote the *distance* and the *angle*, respectively, from the axial plane A_S to the surface S, both provided with a sign according to an orientation of A_S. Then we have (cf. IV.5(3))

$$\tanh \delta = \sin \varphi .$$

Assume $\alpha > 0$, so A_S is an e-hemisphere, and that the latter is oriented towards its exterior. The right-angled e-triangle with a point of the horizon and the e-centers of S and A_S as vertices yields

$$\sin \varphi = \frac{\beta}{\alpha r} = \frac{\beta}{\sqrt{b\bar{b} + \beta^2 - \alpha\gamma}} .$$

For δ we therefore obtain

$$\begin{aligned} \coth \delta &= \frac{\alpha r}{\beta} = \frac{\sqrt{b\bar{b} + \beta^2 - \alpha\gamma}}{\beta}, \\ \sinh \delta &= \frac{\beta}{\sqrt{b\bar{b} - \alpha\gamma}} . \end{aligned} \tag{7}$$

The expressions not containing r are easily seen to be valid if $\alpha = 0$, that is, if S and A_S are e-half-planes, and if A_S is oriented by its normal vector $-b$.

In all three cases the quantity $(\alpha\gamma - b\bar{b})/\beta^2$ which will be denoted by $p(S, S)$, is invariant under the isometry group $\mathscr{H}_3$. We have

$$p(S, S) = \frac{\alpha\gamma - b\bar{b}}{\beta^2} = \begin{cases} 1/\cosh^2 \varrho & \text{for a sphere,} \\ 0 & \text{for a horosphere,} \\ -1/\sinh^2 \delta & \text{for an equidistant surface.} \end{cases} \tag{8}$$

We assume from now on that $\beta \neq 0$, so we exclude the planes.

It will often be convenient to let the spherical surface S be given by the pair,

determined up to a non-zero real factor,

$$(9) \qquad (\mathbf{S}, \beta), \quad \mathbf{S} = \begin{pmatrix} b & -\gamma \\ \alpha & -\bar{b} \end{pmatrix}, \quad \det \mathbf{S} = \alpha\gamma - b\bar{b} = \beta^2 p(S, S),$$

instead of the equation (1)

If S is a sphere, $\mathbf{S}$ is a point matrix since

$$\det \mathbf{S} = \alpha\gamma - b\bar{b} > 0 .$$

It determines the center (4) of S. If $\alpha > 0$, and thus $\beta > 0$, which we may assume, the normalized matrix is

$$(10) \qquad \frac{1}{\sqrt{\det \mathbf{S}}} \mathbf{S} = \frac{1}{\sqrt{\alpha\gamma - b\bar{b}}} \mathbf{S} .$$

If S is a horosphere, $\mathbf{S}$ is singular and determines the improper center b/α of S.

If S is an equidistant surface, $i\mathbf{S}$ is a plane matrix with

$$\det(i\mathbf{S}) = b\bar{b} - \alpha\gamma > 0 .$$

A normalized matrix is

$$(11) \qquad \frac{i}{\sqrt{\det(i\mathbf{S})}} \mathbf{S} = \frac{i}{\sqrt{b\bar{b} - \alpha\gamma}} \mathbf{S} .$$

It determines the axial plane A_S of S with an orientation. According to VII.1, the latter can be read off as follows: If $\alpha \neq 0$, so A_S is an e-hemisphere, the positive half-space is its exterior or interior according as $\alpha > 0$ or $\alpha < 0$. If $\alpha = 0$, so A_S is a vertical e-half-plane, the positive half-space is the one into which $-b$ points. To change the orientation one has to replace $(\alpha, b, \beta, \gamma)$ by $(-\alpha, -b, -\beta, -\gamma)$.

VIII.2 An invariant of a pair of spherical surfaces

Let S and S' be spherical surfaces with equations

$$\begin{aligned} \alpha\,(x\bar{x} + \xi^2) - \bar{b}\,x - b\,\bar{x} - 2\beta\,\xi + \gamma &= 0, \quad \beta \neq 0, \\ \alpha'(x\bar{x} + \xi^2) - \bar{b}'x - b'\bar{x} - 2\beta'\xi + \gamma' &= 0, \quad \beta' \neq 0 . \end{aligned}$$

We claim that the quantity

$$(1) \qquad p(S, S') = \frac{\alpha\gamma' + \gamma\alpha' - b\bar{b}' - \bar{b}b'}{2\beta\beta'}$$

is invariant under $\mathscr{H}_3$.

To see this consider the matrix

$$\frac{1}{\beta}\mathbf{S}\frac{1}{\beta'}\bar{\mathbf{S}}' = \frac{1}{\beta\beta'}\begin{pmatrix} b & -\gamma \\ \alpha & -\bar{b}\end{pmatrix}\begin{pmatrix} \bar{b}' & -\gamma' \\ \alpha' & -b'\end{pmatrix}.$$

It determines a motion, namely the product of two reflections if none of S and S' is a horosphere. Otherwise it determines a singular map sending the whole of U into an improper point. In any case

$$\text{(2)} \qquad \operatorname{tr}\left(\frac{1}{\beta}\mathbf{S}\frac{1}{\beta'}\bar{\mathbf{S}}'\right) = \frac{1}{\beta\beta'}(b\bar{b}' + \bar{b}b' - \alpha\gamma' - \gamma\alpha') = -2p(S, S')$$

is invariant under motions and, since it is real, also under reversals.

For $S = S'$ we obtain the invariant $p(S, S)$ mentioned in the preceding section.

Another special case of interest is that in which one of the surfaces, S' say, degenerates to a proper point $\mathrm{z} = z + \zeta j$. Using VIII.1(4) with $r = 0$, we obtain $b' = \alpha' z$, $\beta' = \alpha'\zeta$. Further VIII.1(2) with $r = 0$ and VIII.1(5) with $\varrho = 0$ yield

$$\gamma' = \frac{1}{\alpha'}(b'\bar{b}' + \beta'^2) = \alpha'(zz' + \zeta^2).$$

Writing here z instead of S', we have

$$\text{(3)} \qquad p(S, \mathrm{z}) = \frac{\alpha(z\bar{z} + \zeta^2) - \bar{b}z - b\bar{z} + \gamma}{2\beta\zeta}.$$

As will be shown in the following section, this quantity is closely related to an analogue of the Euclidean "power of a point with respect to a sphere".

To interprete $p(S, S')$ geometrically we consider first the cases where horospheres do not occur.

Suppose that S and S' are *spheres* with radii $\varrho \geq 0$, $\varrho' \geq 0$, and let $\sigma_{cc'} \geq 0$ denote the distance of their centers c and c$'$. Assuming $\alpha > 0$, $\alpha' > 0$, we apply VII.3(1) to the normalized matrices belonging to S and S' (cf. VIII.1(10). We obtain, using VIII.1(5),

$$\cosh\sigma_{cc'} = -\frac{1}{2}\operatorname{tr}\left(\frac{1}{\sqrt{\alpha\gamma - b\bar{b}}}\mathbf{S}\frac{1}{\sqrt{\alpha'\gamma' - b'\bar{b}'}}\bar{\mathbf{S}}'\right)$$

$$= -\frac{1}{2\beta\beta'}\cosh\varrho\cosh\varrho'\operatorname{tr}(\mathbf{S}\bar{\mathbf{S}}')$$

and hence by (2)

$$\text{(4)} \qquad p(S, S') = \frac{\cosh\sigma_{cc'}}{\cosh\varrho\cosh\varrho'}.$$

Let S be an *equidistant surface* with the axial plane A_S oriented towards S and

with distance $\delta > 0$ from A_S, and let S' be a *sphere* with centre c$'$ and radius $\varrho' \geq 0$. The distance from A_S to c$'$, provided with a sign in accordance with the orientation of A_S, is denoted by $\sigma_{Ac'}$. Applying VII.3(5) to the normalized matrices determining A_S and c$'$ and using VIII.1(7), (5), we obtain

$$\begin{aligned}\sinh \sigma_{Ac'} &= \frac{i}{2}\operatorname{tr}\left(\frac{i}{\sqrt{b\bar{b}-\alpha\gamma}}\mathbf{S}\frac{1}{\sqrt{\alpha'\gamma'-b'\bar{b}'}}\bar{\mathbf{S}}'\right)\\ &= -\frac{1}{2\beta\beta'}\sinh\delta\cosh\varrho'\operatorname{tr}(\mathbf{S}\bar{\mathbf{S}}')\end{aligned}$$

and hence by (2)

$$p(S,S') = \frac{\sinh \sigma_{Ac'}}{\sinh\delta\cosh\varrho'}. \tag{5}$$

Let now S and S' be *equidistant surfaces*. Here we have to distinguish between two cases: 1) the axial planes A_S and $A_{S'}$ are ultraparallel, 2) A_S and $A_{S'}$ intersect, are parallel or coincide.

In the first case we assume the common normal of A_S and $A_{S'}$ to be oriented and the distances δ from A_S to S and δ' from $A_{S'}$ to S' to be provided with signs in accordance with this orientation. Let the normalized matrices determining A_S and $A_{S'}$ (cf. VIII.1(11)) be chosen such that the orientations they determine agree with that of the common normal. If $\sigma_{AA'} > 0$ denotes the distance of A_S and $A_{S'}$, application of VII.3(8) and VIII.1(7) yields

$$\begin{aligned}\cosh \sigma_{AA'} &= \frac{1}{2}\operatorname{tr}\left(\frac{i}{\sqrt{b\bar{b}-\alpha\gamma}}\mathbf{S}\frac{-i}{\sqrt{b'\bar{b}'-\alpha'\gamma'}}\bar{\mathbf{S}}'\right)\\ &= \frac{1}{2\beta\beta'}\sinh\delta\sinh\delta'\operatorname{tr}(\mathbf{S}\bar{\mathbf{S}}')\end{aligned}$$

and hence by (2)

$$p(S,S') = -\frac{\cosh\sigma_{AA'}}{\sinh\delta\sinh\delta'}. \tag{6}$$

In the second case we assume A_S and $A_{S'}$ to be oriented such that the angle $\psi_{AA'}$ between the positive normals at a common (proper or improper) point satisfies $\psi_{AA'} \in [\![0,\pi/2]\!]$. The distances δ and δ' are provided with signs according to these orientations. If the normalized matrices determining A_S and $A_{S'}$ are chosen in accordance with these orientations, application of VII.3(14) and VIII.1(7) yields

$$\begin{aligned}\cos \psi_{AA'} &= \frac{1}{2}\operatorname{tr}\left(\frac{i}{\sqrt{b\bar{b}-\alpha\gamma}}\mathbf{S}\frac{-i}{\sqrt{b'\bar{b}'-\alpha'\gamma'}}\bar{\mathbf{S}}'\right)\\ &= \frac{1}{2\beta\beta'}\sinh\delta\sinh\delta'\operatorname{tr}(\mathbf{S}\bar{\mathbf{S}}')\end{aligned}$$

and hence by (2)

$$p(S,S') = -\frac{\cos\psi_{AA'}}{\sinh\delta\,\sinh\delta'}\,. \tag{7}$$

We consider now the cases where one or both of S and S' are horospheres. Here we have to use a different method.

Let S be a *horosphere*. Because of the invariance of the quantities involved we may assume it to be the horizontal e-plane through j. An equation of it is $\xi - 1 = 0$, hence we may choose $\alpha = 0$, $b = 0$, $\beta = 1$, $\gamma = 2$. Then

$$p(S,S') = \frac{\alpha'}{\beta'}\,. \tag{8}$$

If S' is a *sphere*, we may assume that its center lies on the j-axis. Then we have $b' = 0$, and the center is (cf. VIII.1(4))

$$c' = \sqrt{\frac{\gamma'}{\alpha'}}\,j\,,$$

and for the radius ϱ' we have (cf. VIII.1(5))

$$\cosh\varrho' = \frac{\beta'}{\sqrt{\alpha'\gamma'}}\,.$$

Let $\sigma_{Sc'}$ denote the distance of c' from S, positive or negative according as c' lies in the exterior, here below S, or in the interior of S. Then, assuming the coefficients to be chosen such that $\alpha' > 0$, we have

$$\exp\sigma_{Sc'} = \frac{1}{\sqrt{\gamma'/\alpha'}} = \frac{\alpha'}{\beta'}\cosh\varrho'\,,$$

hence by (8)

$$p(S,S') = \frac{\exp\sigma_{Sc'}}{\cosh\varrho'}\,. \tag{9}$$

If S' is also a *horosphere* and if its center is different from the center ∞ of S, we may assume the center of S' to be 0. The line joining the centers, the j-axis, is the common normal of S and S'. Let $\sigma_{SS'}$ be the distance, measured on it, between S and S', positive if they are disjoint and negative if they intersect. Since S and S' intersect the j-axis at j and $(2\beta'/\alpha')j$, respectively, we have by (8)

$$\exp\sigma_{SS'} = \frac{\alpha'}{2\beta'} = \frac{1}{2}p(S,S')\,,$$

$$p(S,S') = 2\exp\sigma_{SS'}\,. \tag{10}$$

If the horospheres S and S' have the same center, which may be assumed to be ∞, we have

$$p(S, S') = 0$$

since $\alpha = \alpha' = 0$, $b = b' = 0$.

If S' is an *equidistant surface*, two cases have to be distinguished: the horizon of S' does not or does pass through the center of S.

In the first case the diameter of S' ending at the center of S is the common normal of S and S'. Let it be oriented towards the exterior of S. Further, let the distances $\sigma_{SA'}$ from S to the axial plane $A_{S'}$ and δ' from $A_{S'}$ to S' be provided with signs according to this orientation. Assuming as before that S is the horizontal e-plane through j, and that the e-center of $A_{S'}$ is 0, the common normal is the j-axis and the orientation to be used opposite the usual one. Since $A_{S'}$ intersects the j-axis at $\sqrt{-\gamma'/\alpha'}\; j$, we have

$$\exp \sigma_{SA'} = \frac{1}{\sqrt{-\gamma'/\alpha'}} = \frac{\alpha'}{\sqrt{-\alpha'\gamma'}} = \frac{\alpha'}{\beta'} \sinh \delta' ,$$

where the coefficients again are supposed to be chosen such that $\alpha' > 0$, and VIII.1(7) is used. By (8) we therefore obtain

$$(11) \qquad p(S, S') = \frac{\exp \sigma_{SA'}}{\sinh \delta'} .$$

In the second case S' must be an e-half-plane since its horizon has to contain the center ∞ of S. Hence, $\alpha' = 0$ and (8) shows that

$$(12) \qquad p(S, S') = 0 .$$

Finally we are going to prove:

If two spherical surfaces S and S' with equations

$$\alpha\,(x\bar{x} + \xi^2) - \bar{b}\; x - b\; \bar{x} - 2\beta\; \xi + \gamma\; = 0 ,$$
$$\alpha'(x\bar{x} + \xi^2) - \bar{b}'x - b'\bar{x} - 2\beta'\xi + \gamma' = 0$$

intersect, they intersect orthogonally if and only if

$$(13) \qquad p(S, S') = 1 .$$

This is equivalent to

$$(14) \qquad \alpha\gamma' + \gamma\alpha' - b\bar{b}' - \bar{b}b' - 2\beta\beta' = 0 .$$

In the latter form the condition holds also if one or both of S and S' are planes, so $\beta\beta' = 0$.

Suppose first that S as well as S' is an e-sphere or part of one. Let r and r' be their e-radii and s the e-distance of their e-centers. Then the condition for orthogonal intersection is

$$s^2 - r^2 - r'^2 = 0 .$$

Now according to VIII.1(2) we have

$$r^2 = \frac{b\bar{b} + \beta^2 - \alpha\gamma}{\alpha^2}, \quad r'^2 = \frac{b'\bar{b}' + \beta'^2 - \alpha'\gamma'}{\alpha'^2}$$

and, since the e-centers are $(b + \beta j)/\alpha$, $(b' + \beta' j)/\alpha'$,

$$\begin{aligned} s^2 &= \left(\frac{b'}{\alpha'} - \frac{b}{\alpha}\right)\left(\frac{\bar{b}'}{\alpha'} - \frac{\bar{b}}{\alpha}\right) + \left(\frac{\beta'}{\alpha'} - \frac{\beta}{\alpha}\right)^2 \\ &= \frac{b'\bar{b}'}{\alpha'^2} + \frac{b\bar{b}}{\alpha^2} + \frac{\beta'^2}{\alpha'^2} + \frac{\beta^2}{\alpha^2} - \frac{b\bar{b}' + b'\bar{b} + 2\beta\beta'}{\alpha\alpha'} . \end{aligned}$$

Hence

$$s^2 - r^2 - r'^2 = \frac{\alpha\gamma' + \gamma\alpha' - b\bar{b}' - b'\bar{b} - 2\beta\beta'}{\alpha\alpha'},$$

and this proves the statement if $\alpha\alpha' \neq 0$.

If S is an e-plane or e-half-plane, so $\alpha = 0$, the condition is that the e-center of S' lies in S. Hence it is

$$-\bar{b}\frac{b'}{\alpha'} - b\frac{\bar{b}'}{\alpha'} - 2\beta\frac{\beta'}{\alpha'} + \gamma = 0 ,$$

which reduces to (14) with $\alpha = 0$.

If both S and S' are e-planes or e-half-planes, the condition is that the inner product of normal vectors $b + \beta j$ and $b' + \beta' j$ vanishes, and this is easily seen to be (14) with $\alpha = \alpha' = 0$.

It should be observed that (14) is the condition of orthogonality of the e-spheres carrying S and S'. It therefore holds also if these e-spheres intersect orthogonally in the lower half-space.

VIII.3 The power of a point with respect to a spherical surface

As mentioned in the preceding section there is an analogue of the Euclidean notion of the power of a point with respect to a sphere. Let S be a spherical surface and $\mathrm{z} = z + \zeta j$ a proper point through which there is a line L intersecting S twice. If S is a sphere or a horosphere, every point satisfies this requirement. If S

is an equidistant surface, z has to lie on the same side of the axial plane as S. We observe that there is a line through z intersecting S twice if and only if

$$(1) \qquad p(S, \mathrm{z}) > 0$$

(cf. VIII.2(3)). Indeed, VIII.2(4) and (9) with $\varrho' = 0$ and $\mathrm{c}' = \mathrm{z}$ show that (1) holds for all z if S is a sphere or horosphere, and VIII.2(5) with $\varrho' = 0$ and $\mathrm{c}' = \mathrm{z}$ shows that in the case of an equidistance surface S, (1) holds if and only if $\sigma_{Ac'}$ and δ have the same sign.

Choose an orientation of the line L intersecting S, and let τ_1 and τ_2 denote the distances from z to the points of intersection t_1 and t_2 of L and S, provided with signs according to this orientation. We shall prove

$$(2) \qquad \tanh\frac{\tau_1}{2}\tanh\frac{\tau_2}{2} = \frac{p(S, \mathrm{z}) - 1}{p(S, \mathrm{z}) + 1}.$$

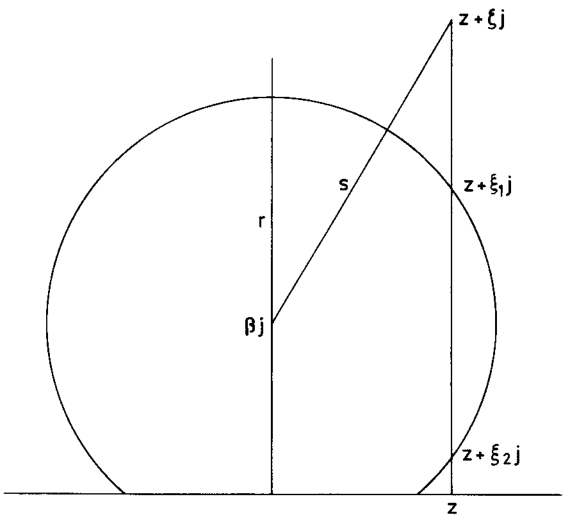
This shows that the left-hand side is independent of the line L through z and may therefore be called the *power* of z with respect to S.

We note that if z lies in the exterior of S, the left-hand side equals $\tanh^2(\tau/2)$, where τ denotes the length of a tangent from z to S.

In the proof of (2) we may assume S to be an e-sphere or part of one with e-center on the j-axis in the upper half-space, further that L is a vertical e-half-line, since all this can be achieved by suitable motions. Coefficients of an equation of S are then $\alpha = 1$, $b = 0$, $\beta > 0$, γ. Hence we have

$$p(S, \mathrm{z}) = \frac{z\bar{z} + \zeta^2 + \gamma}{2\beta\zeta},$$

and the e-center of S is βj.



Let $z + \xi_1 j$, $z + \xi_2 j$ be the points at which L intersects S. The j-coordinates ξ_1 and ξ_2 are the roots of the equation

$$\xi^2 - 2\beta\xi + z\bar{z} + \gamma = 0 .$$

We note that

$$\xi_1 + \xi_2 = 2\beta .$$

For the distances τ_1 and τ_2 from z to the intersection points we have

$$e^{\tau_1} = \zeta/\xi_1 , \quad e^{\tau_2} = \zeta/\xi_2 .$$

The well-known property of the Euclidean power of z with respect to S yields

$$(\zeta - \xi_1)(\zeta - \xi_2) = s^2 - r^2 ,$$

where s denotes the e-distance of z from the e-center of S and r the e-radius of S. The left-hand side equals

$$\zeta^2 (1 - e^{-\tau_1})(1 - e^{-\tau_2})$$

and the right-hand side

$$z\bar{z} + (\zeta - \beta)^2 - (\beta^2 - \gamma) = 2\beta\zeta(p(S, z) - 1) .$$

Hence we have

$$\begin{aligned} p(S,z) - 1 &= \frac{\zeta}{2\beta}(1 - e^{-\tau_1})(1 - e^{-\tau_2}) \\ &= \frac{\zeta}{\xi_1 + \xi_2}(1 - e^{-\tau_1})(1 - e^{-\tau_2}) \\ &= \frac{(1 - e^{-\tau_1})(1 - e^{-\tau_2})}{e^{-\tau_1} + e^{-\tau_2}} . \end{aligned}$$

$$p(S,z) + 1 = \frac{(1 + {}^{-\tau_1})(1 + e^{-\tau_2})}{e^{-\tau_1} + e^{-\tau_2}} .$$

Consequently, we obtain the statement

$$\frac{p(S,z) - 1}{p(S,z) + 1} = \frac{(1 - e^{-\tau_1})(1 - e^{-\tau_2})}{(1 + e^{-\tau_1})(1 + e^{-\tau_2})} = \tanh\frac{\tau_1}{2}\tanh\frac{\tau_2}{2} .$$

VIII.4 The radical plane of a pair of spherical surfaces

Let S and S' be distinct spherical surfaces with equations

$$\alpha\,(x\bar{x}+\xi^2)-\bar{b}\,x-b\,\bar{x}-2\beta\,\xi+\gamma\ =0\,,$$
$$\alpha'(x\bar{x}+\xi^2)-\bar{b}'x-b'\bar{x}-2\beta'\xi+\gamma'=0\,.$$

We are asking for the locus of the points $\mathrm{z}=z+\zeta j$ which have equal powers with respect to S and S'. According to VIII.3(2), a necessary condition z has to satisfy is

(1) $$p(S,\mathrm{z})=p(S',\mathrm{z})\,,$$

explicitly,

$$\frac{\alpha(z\bar{z}+\zeta^2)-\bar{b}z-b\bar{z}+\gamma}{2\beta\zeta}=\frac{\alpha'(z\bar{z}+\zeta^2)-\bar{b}'z-b'\bar{z}+\gamma'}{2\beta'\zeta}$$

which reduces to

(2) $$\left(\frac{\alpha}{\beta}-\frac{\alpha'}{\beta'}\right)(z\bar{z}+\zeta^2)-\left(\frac{\bar{b}}{\beta}-\frac{\bar{b}'}{\beta'}\right)z-\left(\frac{b}{\beta}-\frac{b'}{\beta'}\right)\bar{z}+\frac{\gamma}{\beta}-\frac{\gamma'}{\beta'}=0\,.$$

This is the equation of a plane determined by the plane matrix

(3) $$\begin{pmatrix} b\beta'-b'\beta & -\gamma\beta'+\gamma'\beta \\ \alpha\beta'-\alpha'\beta & -\bar{b}\beta'+\bar{b}'\beta \end{pmatrix} i=(\beta'\mathbf{S}-\beta\mathbf{S}')\,i$$

(cf. VIII.1(9)), provided

$$\det((\beta'\mathbf{S}-\beta\mathbf{S}')\,i)=-\beta'^2\det\mathbf{S}-\beta^2\det\mathbf{S}'-\beta\beta'\operatorname{tr}(\mathbf{S}\bar{\mathbf{S}}')>0\,,$$

where $\mathbf{S}'^{\sim}=-\bar{\mathbf{S}}'$ is used. Because of VIII.1(9) and VIII.2(2) this condition is equivalent to

(4) $$2p(S,S')>p(S,S)+p(S',S')\,.$$

If it is not satisfied, the equation (2) is not satisfied by any proper point.

If the inequality (4) holds, the plane R determined by (2) is called the *radical plane* of the pair S, S' of spherical surfaces. It contains the locus of the points of equal power with respect to S and S'.

Clearly, if S and S' intersect, they have a radical plane, namely the plane containing the intersection. Further, if at least one of S and S' is a sphere or horosphere and they have a radical plane, the whole of it is the locus of the points of equal power since every point has a power with respect to a sphere or horosphere. If two equidistant surfaces have a radical plane, the locus may be a half-plane or even empty, as will be shown below. Concerning the position of the

radical plane we observe: If S and S' have a unique common diameter line (which is the case unless S and S' are concentric spheres or horospheres or equidistant surfaces with non-ultraparallel axial planes), the radial plane is orthogonal to this normal since the configuration is invariant under rotations about it.

We are going to derive criteria for the existence of a radical plane in the various cases.

Assume that S and S' are *spheres*, ϱ and ϱ' their radii and $\sigma_{cc'} \geq 0$ the distance of their centres. Using VIII.1(9) and VIII.2(3), we may write the condition (4) for the existence of a radical plane

$$\cosh \sigma_{cc'} > \frac{1}{2}\left(\frac{\cosh \varrho'}{\cosh \varrho} + \frac{\cosh \varrho}{\cosh \varrho'}\right).$$

Since the function $\xi \mapsto \frac{1}{2}(\xi + 1/\xi)$ is increasing for $\xi \geq 1$, this inequality is equivalent to

$$\exp \sigma_{cc'} > \max\left\{\frac{\cosh \varrho'}{\cosh \varrho}, \frac{\cosh \varrho}{\cosh \varrho'}\right\},$$

$$\sigma_{cc'} > |\log \cosh \varrho' - \log \cosh \varrho| . \tag{5}$$

Hence, if $\varrho \neq \varrho'$, there is a positive lower bound for the distance of the centers. It is easily seen that, if the spheres do not intersect, the radical plane lies between them if they lie outside each other, and it lies outside the larger one if one lies in the interior of the other.

Let S be a *horosphere* and S' a *sphere* with radius ϱ', and let $\sigma_{Sc'}$ again denote the distance from S to the center c′ of S', positive or negative according as c′ lies in the exterior or interior of S. By VIII.1(9) and VIII.2(4) the condition (4) is here

$$\exp \sigma_{Sc'} > \frac{1}{2 \cosh \varrho'},$$

$$\sigma_{Sc'} > -\log (2 \cosh \varrho'), \tag{6}$$

so there is a negative lower bound for $\sigma_{Sc'}$. Concerning the position of the radical plane, the same can be said as in the case of two spheres, with S taking the role of the larger sphere.

Let S and S' be *horospheres* and $\sigma_{SS'}$ their distance. If they have different centers, VIII.1(9) and VIII.2(5) show that (4) is here

$$\exp \sigma_{SS'} > 0,$$

so always satisfied. Hence, any two horospheres with different centers have a radical plane. Its position relative to S and S' is as in the case of two spheres. If S and S' have the same center, (4) would require $0 > 0$ because of VIII.1(9) and VIII.2(6). Hence, two concentric horospheres have no radical plane.

Let S be an *equidistant surface* with distance $\delta > 0$ from its axial plane A_S and S' a *sphere* with radius ϱ' the center c′ of which has the distance $\sigma_{Ac'}$ from A_S, positive or negative according as c′ lies on the same or the opposite side of A_S as S. Here VIII.1(9) and VIII.2(7) show that (4) is equivalent to

$$\sinh \sigma_{Ac'} > \frac{1}{2}\left(\frac{\sinh \delta}{\cosh \varrho'} - \frac{\cosh \varrho'}{\sinh \delta}\right).$$

Observing that the function $\xi \mapsto \frac{1}{2}(\xi - 1/\xi)$ is increasing for $\xi > 0$, we may replace the inequality by

$$\exp \sigma_{Ac'} > \frac{\sinh \delta}{\cosh \varrho'},$$

$$\sigma_{Ac'} > \log \sinh \delta - \log \cosh \varrho'. \tag{7}$$

Hence, there is a (positive or negative) lower bound for the distance from A_S to c′. Statements about the position of the radical plane are again obvious.

If S is an *equidistant surface* and S' a *horosphere*, as in the previous section, three cases have to be distinguished: the center of S' is an improper point 1) of the exterior of S, 2) of the interior of S, 3) of the horizon of S. In cases 1) and 2) S and S' have a common diameter supposed to be oriented towards the exterior of S. The distance $\varrho_{AS'}$ from the axial plane A_S to S' is provided with a sign accordingly, and we have $\delta > 0$.

In case 1), VIII.1(9) and VIII.2(8) imply that (4) is here

$$\exp \sigma_{AS'} > -\frac{1}{2 \sinh \delta},$$

hence always satisfied. In case 2) there is also always a radical plane since S and S' necessarily intereintersect. In case 3) the condition (4) is

$$-\frac{2 \exp(-\sigma_{AS'})}{\sinh \delta} > -\frac{1}{\sinh^2 \delta}$$

because of VIII.1(9) and VIII.2(9). Since $\delta > 0$, it reduces to

$$\exp \sigma_{AS'} > 2 \sinh \delta,$$

$$\sigma_{AS'} > \log(2 \sinh \delta). \tag{8}$$

Hence there is a (positive or negative) lower bound for the distance of S' from the axial plane A_S.

As mentioned before, in all the cases considered so far, the entire radical plane, if it exists, is the locus of the points of equal power with respect to S and S'. If both surfaces are *equidistant surfaces* the situation is more involved.

Suppose first that the axial planes A_S and $A_{S'}$ are *ultraparallel*. Recall that the

distances δ from A_S to S and δ' from $A_{S'}$ to S' are supposed to be provided with signs according to the orientation of the common normal of A_S and $A_{S'}$ from A_S towards $A_{S'}$. As before, $\sigma_{AA'} > 0$ denotes the distance from A_S to $A_{S'}$. Because of VIII.1(9) and VIII.2(11) the condition (4) is here

$$(9) \qquad -\frac{2\cos\sigma_{AA'}}{\sinh\delta\sinh\delta'} > -\frac{1}{\sinh^2\delta} - \frac{1}{\sinh^2\delta'} .$$

If $\delta\delta' > 0$, it may be written

$$\cosh\sigma_{AA'} < \frac{1}{2}\left(\frac{\sinh\delta'}{\sinh\delta} + \frac{\sinh\delta}{\sinh\delta'}\right),$$

thus, by an argument used above,

$$\exp\sigma_{AA'} < \max\left\{\frac{\sinh\delta'}{\sinh\delta}, \frac{\sinh\delta}{\sinh\delta'}\right\},$$

$$(10) \qquad \sigma_{AA'} < \left|\log\sinh|\delta'| - \log\sinh|\delta|\right| .$$

If $\delta = \delta'$, this is not satisfied, so there is no radical plane. Otherwise there is an upper bound for the distance of the axial planes. If $\delta\delta' < 0$, (9) is always satisfied, so there exists a radical plane for all $\sigma_{AA'} > 0$.

To obtain information about the position of the radical plane R and about the locus of points of equal power, we use equations of A_S, $A_{S'}$, S, S'. We may assume that the common normal of A_S and $A_{S'}$ is the j-axis with its usual orientation, so that $A_{S'}$ lies above A_S. The planes A_S, $A_{S'}$, and R have the common e-center 0. Let $a > 0$ and $a' > a$ denote the e-radii of A_S and $A_{S'}$. Then equations of these planes may be written

$$x\bar{x} + \xi^2 - a^2 = 0 , \qquad x\bar{x} + \xi^2 - a'^2 = 0$$

and equations for S and S' are

$$x\bar{x} + \xi^2 - 2\beta\xi - a^2 = 0 , \quad x\bar{x} + \xi^2 - 2\beta'\xi - a'^2 = 0 .$$

Since the coefficients in (1) are here

$$\alpha = 1 , \ b = 0 , \ \gamma = -a^2 ; \ \alpha' = 1 , \ b' = 0 , \ \gamma' = -a'^2 ,$$

VIII.1(8) yields

$$(11) \qquad \sinh\delta = \beta/a , \qquad \sinh\delta' = \beta'/a' .$$

Further we have

$$(12) \qquad \exp\sigma_{AA'} = a'/a$$

since $\sigma_{AA'}$ is the distance of the points aj and $a'j$.

According to (2), an equation of the radical plane R is

$$(\beta' - \beta)(z\bar{z} + \zeta^2) - (\beta' a^2 - \beta a'^2) = 0$$

and the condition for its existence

$$(\beta' - \beta)(\beta' a^2 - \beta a'^2) > 0$$

is assumed to be satisfied. For its e-radius r we have

$$r^2 = \frac{\beta' a^2 - \beta a'^2}{\beta' - \beta}$$

$$(13) \qquad = a^2 - \frac{\beta(a'^2 - a^2)}{\beta' - \beta} = a'^2 - \frac{\beta'(a'^2 - a^2)}{\beta' - \beta}.$$

The two last expressions show that the position of R relative to the axial planes A_S and $A_{S'}$ depends on the signs of $\beta/(\beta' - \beta)$ and $\beta'/(\beta' - \beta)$. Since $\beta/(\beta' - \beta) < \beta'/(\beta' - \beta)$, (13) shows that there are three possibilities:

(a) $\beta/(\beta' - \beta) > 0$. Then $r^2 < a^2$, so A_S separates R and $A_{S'}$.

(b) $\beta'/(\beta' - \beta) < 0$. Then $r^2 > a'^2$, so $A_{S'}$ separates R and A_S.

(c) $\beta/(\beta' - \beta) < 0$, $\beta'/(\beta' - \beta) > 0$. Then $a^2 < r^2 < a'^2$, so R separates A_S and $A_{S'}$.

Using (11) and (12), we obtain

$$\frac{\beta}{\beta' - \beta} = \left(\exp \sigma_{AA'} \frac{\sinh \delta'}{\sinh \delta} - 1\right)^{-1},$$

$$\frac{\beta'}{\beta' - \beta} = \left(1 - \exp(-\sigma_{AA'}) \frac{\sinh \delta}{\sinh \delta'}\right)^{-1}.$$

If $\delta\delta' > 0$ and $\sinh \delta'/\sinh \delta > 1$, we have case (a), so R lies on the negative side of A_S. Hence, R is the locus of points of equal power, provided $\delta < 0$, $\delta' < 0$. If $\delta > 0$, $\delta' > 0$, this locus is empty. Analogously, if $\delta\delta' > 0$ and $\sinh \delta'/\sinh \delta < 1$, we have case (b), so R lies on the positive side of $A_{S'}$. Hence, R is the locus of points of equal power, provided $\delta > 0$, $\delta' > 0$. If $\delta < 0$, $\delta' < 0$, the locus is empty. If $\delta\delta' < 0$, we have case (c), so R lies between A_S and $A_{S'}$. It is the locus of points of equal power, provided $\delta > 0$ and $\delta' < 0$, that is, if S lies on the same side of A_S as R and S' on the same side of $A_{S'}$ as R. If $\delta < 0$ and $\delta' > 0$, that is, if S and S' lie outside the region bounded by A_S and $A_{S'}$, the locus is empty.

For the sake of completeness we mention the trivial case $A_S = A_{S'}$. For $\sigma_{AA'} = 0$ the condition (9) requires only $\delta \neq \delta'$. If it is satisfied, the radical plane coincides with $A_S = A_{S'}$. Consequently the locus of points of equal power is empty.

Suppose now that A_S and $A_{S'}$ are *parallel*. The angle $\psi_{AA'}$ introduced in VIII.2

equals 0, and hence VIII.2(12) and VIII.1(9) show that (4) also here only requires $\delta \neq \delta'$. In order to obtain information about the position of the radical plane R we assume A_S and $A_{S'}$ to be the vertical e-half-plane bounded by the imaginary axis and its e-parallel through the point 1, both oriented in accordance with the real axis. If φ and φ' denote the angles from A_S to S and from $A_{S'}$ to S', respectively, provided with signs in the obvious manner, then S and S' are e-half-planes with slopes $\cot \varphi$ and $\cot \varphi'$. Since

$$\tan \varphi = \sinh \delta \,, \qquad \tan \varphi' = \sinh \delta'$$

by IV.5(4), equations of S and S' may be written

$$x + \bar{x} - 2\xi \sinh \delta = 0 \,, \quad x + \bar{x} - 2\xi \sinh \delta' - 2 = 0 \,.$$

In the notation of (1) we have here

$$\begin{aligned} \alpha &= 0 \,, & b &= -1 \,, & \beta &= \sinh \delta \,, & \gamma &= 0 \,, \\ \alpha' &= 0 \,, & b' &= -1 \,, & \beta' &= \sinh \delta' , & \gamma' &= -2 \,, \end{aligned}$$

so the equation (2) of R may be written

$$z + \bar{z} = \frac{2 \sinh \delta}{\sinh \delta - \sinh \delta'} \,.$$

The right-hand side is positive and less than 2 if $\delta\delta' < 0$. Hence R lies between A_S and $A_{S'}$ in this case. If $\delta > 0$, $\delta' < 0$, S and S' intersect and R is the locus of points of equal power. If $\delta < 0$, $\delta' > 0$, S and S' lie outside the region bounded by A_S and $A_{S'}$ and the locus is empty. If $\delta > \delta' > 0$ or $\delta' < \delta < 0$, S and S' intersect in the positive half-space of $A_{S'}$ or the negative half-space of A_S, respectively, and R is the locus of points of equal power. In the remaining cases $\delta' > \delta > 0$ and $\delta < \delta' < 0$ the surfaces S and S' are disjoint and lie in the positive half-space of A_S and the negative half-space of $A_{S'}$, respectively, while R lies in the complementary half-spaces, and the locus is empty.

If A_S and $A_{S'}$ intersect, so do S and S' and there exists always a radical plane. That (4) is satisfied may also be checked easily by means of VIII.1(9) and VIII.2(12). recalling that the orientations of A_S and $A_{S'}$ were chosen such that the angle $\psi_{AA'}$ between the positive normals is acute or right. The plane containing the intersection $S \cap S'$ is the radical plane R. It contains also the line $A_S \cap A_{S'}$. To see this assume that S and S' are e-half-planes and, thus, A_S and $A_{S'}$ vertical e-half-planes bounded by the same e-lines. The intersections $S \cap S'$ and $A_S \cap A_{S'}$ are then e-half-lines with common endpoint and therefore contained in the same e-half-plane, vertical since $A_S \cap A_{S'}$ is vertical. The locus of the points of equal power is that open half-plane of R which is bounded by $A_S \cap A_{S'}$ and contains $S \cap S'$. The other half-plane lies in the intersection of those half-spaces bounded by A_S and $A_{S'}$ which do not contain S and S', respectively, so its points have no power with respect to S and S'.

VIII.5 Linear families of spherical surfaces

Let

$$\alpha_\nu(x\bar{x}+\xi^2)-\bar{b}_\nu x-b_\nu\bar{x}-2\beta_\nu\xi+\gamma_\nu=0\,,\quad \nu=1,2,\ldots,n\,,$$

be equations of e-spheres or e-planes S_ν in the space $\mathbb{J}$. We assume that the left-hand sides are linearly independent over $\mathbb{R}$, so $n\le 5$. For real numbers λ_ν, not all zero, we consider the linear combination

$$\alpha(x\bar{x}+\xi^2)-\bar{b}x-b\bar{x}-2\beta\xi+\gamma$$
$$=\sum_{\nu=1}^{n}\lambda_\nu\left[\alpha_\nu(x\bar{x}+\xi^2)-\bar{b}_\nu x-b_\nu\bar{x}-2\beta_\nu\xi+\gamma_\nu\right]=0\,.$$

This determines an e-sphere S and we write

(1) $$S=\lambda_1S_1+\lambda_2S_2+\ldots+\lambda_nS_n\,,$$

provided the condition VIII.1(3), that is,

(2) $$b\bar{b}+\beta^2-\alpha\gamma\ge 0$$

is satisfied. The e-spheres S obtained for varying λ_ν are said to form a *linear family of* e-spheres. The generating e-spheres S_ν may be replaced by any n linearly independent e-spheres of the family without changing the latter. If $n=5$, the family consists of all e-spheres. Hence, only the values $n\le 4$ are of interest.

Because of the homogenity, (1) determines a zero-parameter family, that is, a single e-sphere, if $n=1$, a one-parameter family, a *pencil* if $n=2$, a two-parameter family, a *bundle*, if $n=3$, and a three-parameter family, sometimes called a *net*, if $n=4$.

The left-hand side of (2) is a quadratic form in the λ_ν. The family (1) is called *elliptic, parabolic*, or *hyperbolic*, according as the form is positive definite, positive semidefinite, or indefinite.

The definition above of a linear family of e-spheres may be taken as a definition of a *linear family of spherical surfaces* in hyperbolic space if instead of the e-spheres involved their intersections with the upper half-space U, likewise denoted by S_ν, S, are considered. This means that if $f\in\mathscr{H}_3$ is a hyperbolic isometry and $f(S_\nu), f(S)$ denote the images of the S_ν, S, we have

(3) $$f\Big(\sum_{\nu=1}^{n}\lambda_\nu S_\nu\Big)=\sum_{\nu=1}^{n}\lambda_\nu f(S_\nu)\,.$$

That this holds for $f\colon \mathrm{x}\mapsto \mathrm{x}+q$, $q\in\mathbb{C}$, and $f\colon \mathrm{x}\mapsto -\bar{\mathrm{x}}$ is obvious since these f are e-isometries. They and the anti-inversion

$$f\colon \mathrm{x}\mapsto -\mathrm{x}^{-1}=(-\bar{x}+\xi j)/(x\bar{x}+\xi^2)$$

generate $\mathscr{H}_3$. Therefore it remains to verify that (3) holds for the latter. This follows immediately from the fact that an equation of $f(S_\nu)$ may be written

$$\gamma_\nu(x\bar{x}+\xi^2)+b_\nu x+\bar{b}_\nu\bar{x}-2\beta_\nu\xi+\alpha_\nu=0\,.$$

We shall frequently use the pairs

$$(\mathbf{S}_\nu,\beta_\nu)\,,\qquad (\mathbf{S},\beta)\,,$$

where

$$\mathbf{S}_\nu=\begin{pmatrix} b_\nu & -\gamma_\nu \\ \alpha_\nu & -\bar{b}_\nu \end{pmatrix},\qquad \mathbf{S}=\begin{pmatrix} b & -\gamma \\ \alpha & -\bar{b} \end{pmatrix},$$

to determine S_ν, S. We then have

$$(4)\qquad (\mathbf{S},\beta)=\left(\sum_{\nu=1}^{n}\lambda_\nu\mathbf{S}_\nu\,,\ \sum_{\nu=1}^{n}\lambda_\nu\beta_\nu\right).$$

Let S_ν, $\nu=1,\dots,n$, $1\le n\le 4$, be spherical surfaces with linearly independent equations, and let S now denote a spherical surface which intersects all S_ν orthogonally. According to VIII.2(14),

$$(5)\qquad \alpha\gamma_\nu+\gamma\alpha_\nu-b\bar{b}_\nu-\bar{b}b_\nu-2\beta\beta_\nu=0\,,\quad \nu=1,\dots,n\,.$$

This shows that the coefficients α, b, β, γ also satisfy every linear combination with real coefficients of these equations. It means that all the spherical surfaces of the linear family generated by the S_ν are intersected orthogonally by S.

Since (5) for given S_ν may be considered as a linearly independent system of homogeneous linear equations with unknowns α, $\operatorname{Re} b$, $\operatorname{Im} b$, β, γ, the spherical surfaces S_ν intersecting the S_ν orthogonally form a linear $(4-n)$-parameter family. In particular, the spherical surfaces orthogonal to those of a pencil form a bundle, and conversely.

Next we consider the radical planes of pairs of spherical surfaces belonging to a linear family.

Let $S_1, S_2, \dots, S_n$, $n=2,3,4$, be linearly independent spherical surfaces any two of which have a radical plane. Then these radical planes belong to an $(n-2)$-parameter linear family of planes, and the radical plane of any two of the spherical surfaces of the linear family generated by the S_ν belongs to the same linear family of planes.

For $n=2$ this statement is: If S_1 and S_2 have a radical plane, any two spherical surfaces of the pencil generated by S_1 and S_2 have the same radical plane.

To prove this we recall that the radical plane of S_1 and S_2 according to VIII.4(3) is determined by the plane matrix $(\beta_2\mathbf{S}_1-\beta_1\mathbf{S}_2)i$. It is sufficient to show that, if one of S_1, S_2 is replaced by an arbitrary spherical surface S of the pencil, the same radical plane is obtained. Let S be determined by

$$(\mathbf{S},\beta)=(\lambda_1\mathbf{S}_1+\lambda_2\mathbf{S}_2,\lambda_1\beta_1+\lambda_2\beta_2)\,.$$

A plane matrix determining the radical plane of S_1 and S is then

$$[(\lambda_1\beta_1+\lambda_2\beta_2)\mathbf{S}_1-\beta_1(\lambda_1\mathbf{S}_1+\lambda_2\mathbf{S}_2)]\,i=\lambda_2(\beta_2\mathbf{S}_1-\beta_1\mathbf{S}_2)\,i\,,$$

and this proves the statement.

For $n=3$ the statement is: If any two of S_1, S_2, S_3 have a radical plane, these planes belong to a pencil of planes, and the radical plane of any two spherical surfaces of the bundle generated by S_1, S_2, S_3 belongs to the same plane pencil.

To show this we observe that the plane matrices determining the three radical planes of S_1, S_2, S_3 are linearly dependent; indeed,

$$(6)\qquad \beta_3(\beta_2\mathbf{S}_1-\beta_1\mathbf{S}_2)+\beta_1(\beta_3\mathbf{S}_2-\beta_2\mathbf{S}_3)+\beta_2(\beta_1\mathbf{S}_3-\beta_3\mathbf{S}_1)=\mathbf{0}\,.$$

This proves the first part of the statement. The second part follows by repeated application of the following fact: The radical plane of one of S_1, S_2, S_3 and an arbitrary spherical surface S of the bundle generated by S_1, S_2, S_3 belongs to the plane pencil. Consider for instance the radical plane of S_1 and

$$S=\lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3\,.$$

The matrix determining it is indeed

$$\begin{aligned}&[(\lambda_1\beta_1+\lambda_2\beta_2+\lambda_3\beta_3)\mathbf{S}_1-\beta_1(\lambda_1\mathbf{S}_1+\lambda_2\mathbf{S}_2+\lambda_3\mathbf{S}_3)]\,i\\&=[\lambda_2(\beta_2\mathbf{S}_1-\beta_1\mathbf{S}_2)+\lambda_3(\beta_3\mathbf{S}_1-\beta_1\mathbf{S}_3)]\,i\,.\end{aligned}$$

Application hereof to S_1, S_2, S instead of S_1, S_2, S_3 with S_1 replaced by an arbitrary linear combination of S_1, S_2, S yields the general statement.

For $n=4$ the statement is: If any two of $S_1, \ldots, S_4$ have a radical plane, these planes belong to a bundle of planes, and the radical plane of any two spherical surfaces of the net generated by $S_1, \ldots, S_4$ belongs to the same plane bundle.

The spherical surfaces $S_1, \ldots, S_4$ have 6 radical planes which may be denoted by R_{12}, R_{13}, R_{14}, R_{23}, R_{24}, R_{34} in the obvious manner. According to (6) and analogues, R_{23}, R_{24}, R_{34} are linearly dependent on R_{12} and R_{13}, on R_{12} and R_{14}, on R_{13} and R_{14}, respectively. Hence all the six planes belong to the bundle generated by R_{12}, R_{13}, R_{14}, and this is the first part of the statement. That the second part holds can be seen as in the case $n=3$ by replacing S_4, say, by an arbitrary spherical surface of the net, applying the preceding result, and repeating the procedure.

We add some further remarks on radical planes.

If a plane is contained in a linear family of spherical surfaces, it is the radical plane of a pencil of spherical surfaces contained in the family. Indeed, the plane is a linear combination $\lambda S+\lambda' S'$ of spherical surfaces S and S' of the family and thus determined by the pair

$$(\lambda\mathbf{S}+\lambda'\mathbf{S}'\,,\lambda\beta+\lambda'\beta')$$

with $\lambda\beta+\lambda'\beta'=0$, equivalently $(\beta'\mathbf{S}-\beta\mathbf{S}',0)\,.$

Further we note that the part in the upper half-space of the radical e-plane in the Euclidean sense of the e-spheres containing the spherical surfaces S_1 and S_2 is a spherical surface belonging to the pencil generated by S_1 and S_2. But from the hyperbolic point of view it is indistinguishable from the other surfaces of the pencil unless it is a vertical e-half-plane, thus also the radical plane of the pencil in the hyperbolic sense.

To obtain a classification of the linear families of spherical surfaces it is convenient to start with the Euclidean one. The notions of elliptic, parabolic and hyperbolic families defined above make also sense in hyperbolic geometry.

An *elliptic pencil* of e-spheres consists of all e-spheres and the e-plane passing through a fixed e-circle C. To interprete this in hyperbolic geometry one has to distinguish several cases according to the position of C relative to U and $\mathbb{C}_\infty$. If $C \cap U \neq 0$, the pencil consists of the spherical surfaces passing through $C \cap U$. If $C \subset \mathbb{C}_\infty$, the pencil consists of the equidistant surfaces which have C as common horizon and hence a common axial plane. In the remaining cases the pencil consists of equidistant surfaces whose mutual relations in terms of hyperbolic geometry seem to be rather involved. Common to all cases is however that the pencil contains one plane which according to a previous observation is the radical plane. Indeed, whatever the position of C, there is a unique e-sphere or e-plane through C which intersects $\mathbb{C}_\infty$ orthogonally.

An *elliptic bundle* in the Euclidean sense consists of the e-spheres passing through two given distinct points c_1 and c_2. In the hyperbolic sense again various cases have to be distinguished. If at least one of c_1 and c_2 lies in the lower half-space, a description of the bundle in hyperbolic terms seems to be difficult. In all cases the bundle has an elliptic pencil of radical planes. Indeed, there is a unique e-circle or e-line passing through c_1 and c_2 and intersecting $\mathbb{C}_\infty$ orthogonally. The part of it in U is a line, and all planes containing it belong to the bundle.

A *parabolic pencil* of e-spheres consists of the e-spheres and the e-plane touching a given e-sphere S^* at a given point c of it. If c does not lie on $\mathbb{C}_\infty$, there is a unique e-sphere or e-plane in the pencil which intersects $\mathbb{C}_\infty$ orthogonally. The part of it in U is a plane, and hence the radical plane of the pencil in the hyperbolic sense. If c lies in the lower half-space, the pencil consists of mutually disjoint equidistant surfaces. To describe in hyperbolic terms how they are related seems to be difficult. If $c \in \mathbb{C}_\infty$ and S^* is not orthogonal to $\mathbb{C}_\infty$, the pencil consists of the equidistant surfaces which touch S^* at the improper point c. There is no radical plane in this case. If $c \in \mathbb{C}_\infty$ and S^* is orthogonal to $\mathbb{C}_\infty$, the pencil is a parabolic plane pencil.

A *parabolic bundle* of e-spheres consists of the e-spheres which touch a given e-circle C at a given point c of it. If c does not lie on $\mathbb{C}_\infty$, there is a unique e-circle or e-line which touches C at c and intersects $\mathbb{C}_\infty$ orthogonally. The part in U of it is a line, and all planes containing it belong to the bundle. Consequently the latter has an elliptic pencil of radical planes. If c lies in the lower half-space, the description of the bundle in hyperbolic terms is also difficult here. If $c \in \mathbb{C}_\infty$, one has a

parabolic plane bundle if C is orthogonal to $\mathbb{C}_\infty$, otherwise the bundle consists of the equidistant surfaces touching C at the improper point c, and there are no radical planes.

It is easily seen that the spherical surfaces which intersect all of those of a parabolic pencil form a parabolic bundle, and conversely.

A *hyperbolic pencil* of e-spheres consists of the e-spheres which intersect those of an elliptic bundle orthogonally. The e-spheres of the pencil are mutually disjoint, and two of them are point-spheres, namely the points c_1 and c_2 through which the e-spheres of the elliptic bundle pass. These points are inverse with respect to each of the e-spheres of the pencil. Concerning the hyperbolic interpretation we observe that if both c_1 and c_2 lie on $\mathbb{C}_\infty$, the pencil consists of the planes orthogonal to the line $[c_1, c_2]$, so it is a hyperbolic plane pencil. Suppose that not both of c_1 and c_2 lie on $\mathbb{C}_\infty$. We shall show that the pencil has a radical plane if and only if c_1 and c_2 lie in the same open half-space bounded by $\mathbb{C}_\infty$. By means of a suitable motion extended to the lower half-space we can achieve that c_1 and c_2 lie on the j-axis. Let $c_1 = \beta_1 j$, $c_2 = \beta_2 j$. Equations for the point-spheres, which generate the pencil, may then be written

$$x\bar{x} + \xi^2 - 2\beta_1\xi + \beta_1^2 = 0\,, \quad x\bar{x} + \xi^2 - 2\beta_2\xi + \beta_2^2 = 0\,.$$

According to VIII.4(3), they have a radical plane if and only if

$$\det\begin{pmatrix} 0 & (-\beta_1^2\beta_2 + \beta_2^2\beta_1)i \\ (\beta_2 - \beta_1)i & 0 \end{pmatrix} = \beta_1\beta_2(\beta_2 - \beta_1)^2 > 0\,,$$

and this proves the statement.

As is well known, there is a simple e-geometric characterization of the e-spheres of a hyperbolic pencil by means of Apollonius' Theorem: The locus of a point whose e-distances from c_1 and c_2 have a constant ratio is an e-sphere, the e-center of which lies on the e-line joining c_1 and c_2. For the various values of the ratio precisely the e-spheres of the pencil are obtained. If c_1 and c_2 lie in U, the spherical surfaces of the pencil admit of an analogous characterization in hyperbolic geometry. For the sake of simplicity we may again assume that $c_1 = \gamma_1 j$, $c_2 = \gamma_2 j$, here with $\gamma_1 > 0$, $\gamma_2 > 0$. Let σ_1 and σ_2 denote the distances of an arbitrary point $\mathrm{x} = x + \xi j$ from c_1 and c_2. According to III.4(1) we then have

$$\sinh^2\frac{\sigma_1}{2} = \frac{|x + \xi j - \gamma_1 j|^2}{4\gamma_1\xi}\,, \quad \sinh^2\frac{\sigma_2}{2} = \frac{|x + \xi j - \gamma_2 j|^2}{4\gamma_2\xi}\,.$$

Since the numerators of the right-hand sides are the squares of the e-distances of x from c_1 and c_2, their ratio is constant if and only if

$$\sinh^2\frac{\sigma_1}{2} \Big/ \sinh^2\frac{\sigma_2}{2} = \text{const.}$$

Hence, the spherical surfaces of the pencil are the loci of the points satisfying such

a condition. If not both of c_1 and c_2 lie in U, there seems to be no similar property of the spherical surfaces of the pencil.

A *hyperbolic bundle* of e-spheres consists of the e-spheres which intersect all e-spheres of an elliptic pencil orthogonally. The points of the e-circle C through which the e-spheres of the pencil pass are the point-spheres belonging to the bundle. If $\mathrm{c}_1 = c_1 + \gamma_1 j$, $\mathrm{c}_2 = c_2 + \gamma_2 j$, and $\mathrm{c}_3 = c_3 + \gamma_3 j$ are any three distinct point spheres on C, the bundle is generated by them. An equation of an e-sphere of the bundle may therefore be written

$$\lambda_1 |\mathrm{x} - \mathrm{c}_1|^2 + \lambda_2 |\mathrm{x} - \mathrm{c}_2|^2 + \lambda_3 |\mathrm{x} - \mathrm{c}_3|^2 = 0\,,$$

where $\lambda_1, \lambda_2, \lambda_3$ are real numbers satisfying the inequality following from VIII.1(3). Hence, every e-sphere of the bundle is the locus of the points x for which the squares of its e-distances from c_1, c_2 and c_3 satisfy a homogeneous linear relation. If c_1, c_2, c_3 lie in U, VIII.4(1) can be used to translate this to hyperbolic geometry. Indeed the relation above is equivalent to

$$\lambda_1 \gamma_1 \sin^2 \frac{\sigma_1}{2} + \lambda_2 \gamma_2 \sinh^2 \frac{\sigma_2}{2} + \lambda_3 \gamma_3 \sinh^2 \frac{\sigma_3}{3} = 0\,,$$

where σ_1, σ_2, σ_3 denote the distances of x from c_1, c_2, c_3, respectively. The spherical surfaces of the bundle can therefore be determined by such relations provided $C \cap U \neq 0$. If $C \cap U = 0$, there seems to be no similar characterization of them.

We now discuss the existence of radical planes in a hyperbolic bundle of spherical surfaces. Assume first that the circle C, through which the spherical surfaces of the elliptic pencil pass, lies in U. Then the normal N to the plane containing C through the center of C contains the centres of the spheres and horospheres through C, and it intersects the axial planes of the equidistant surfaces through C orthogonally. Hence, every plane through N intersects the spherical surfaces of the elliptic pencil orthogonally, so the bundle has an elliptic pencil of radical planes. The same holds if C lies in the lower half-space. This is most easily seen by reflecting the whole configuration in $\mathbb{C}_\infty$. However, a characterization of the axis N of the pencil in the hyperbolic sense is not obtained herewith. – If C intersects $\mathbb{C}_\infty$, the part of it in U is an equidistant curve. Every plane which intersects its axis A orthogonally, intersects also all spherical surfaces through C orthogonally. Hence, the planes orthogonal to A form a hyperbolic pencil of radical planes of the bundle. It should be observed that in this case not every pencil contained in the bundle has a radical plane. As shown above, this holds for the pencils generated by two point-spheres on C which lie on opposite sides of $\mathbb{C}_\infty$. – If C touches $\mathbb{C}_\infty$ from above, it is a horocycle. The planes which pass through its diameters and are orthogonal to its plane belong to the bundle. They are the radical planes of the latter and form a parabolic plane pencil. If C touches $\mathbb{C}_\infty$ from below, the bundle has also a parabolic pencil of radical planes. This may be seen by reflect-

ing the configuration in $\mathbb{C}_\infty$. – If $C \subset \mathbb{C}_\infty$, the elliptic pencil of spherical surfaces consists of the equidistant surfaces which have the plane with horizon C as common axial plane. The hyperbolic bundle in question is therefore here the hyperbolic bundle of planes orthogonal to the axial plane.

Finally we consider the three-parameter linear families, the *nets*. In the Euclidean sense, an elliptic net consists of the e-spheres which intersect a given e-sphere in great circles of the latter, a parabolic net consists of the e-spheres passing through a given point, and a hyperbolic net consists of the e-spheres intersecting a given e-sphere orthogonally.

We start with the *hyperbolic nets* since their definition can be carried over to hyperbolic geometry provided the given e-sphere S^* lies totally or partly in U. If $S^* \subset U$, the planes through the center c^* of S^* belong to the net. Hence, the latter has an elliptic bundle of radical planes. Since the e-spheres carrying these planes also pass through the mirror image of c^* in $\mathbb{C}_\infty$, reflection in $\mathbb{C}_\infty$ shows that the net determined by an e-sphere S^* in the lower half-space also has an elliptic bundle of radical planes. In this case the net consists of equidistant surfaces whose mutual relations in the hyperbolic sense seem to be involved. – If S^* touches $\mathbb{C}_\infty$ from above or below, the net contains all planes whose horizons pass through the point of contact. They form a parabolic bundle of radical planes. – If the e-sphere S^* intersects $\mathbb{C}_\infty$, the part in U is an equidistant surface. All planes orthogonal to its axial plane belong to the net, so the latter has a hyperbolic bundle of radical planes.

As mentioned above, a *parabolic net* in the e-sense consists of the e-spheres passing through a given point c^*. If $c^* \in U$, this makes also sense in hyperbolic geometry. The planes through c^* form an elliptic bundle of radical planes. If c^* lies in the lower half-space, reflection in $\mathbb{C}_\infty$ gives again some information. In particular, one sees that there is also an elliptic bundle of radical planes, namely the one consisting of the planes through the mirror image of c^*. If $c^* \in \mathbb{C}_\infty$, the net consists of the equidistant surfaces and planes the horizons of which pass through c^* and the horospheres with center c^*. These planes form a parabolic bundle of radical planes.

The Euclidean definition of an *elliptic net* as the set of e-spheres cutting a given e-sphere in great circles makes no sense in hyperbolic geometry. However, it will turn out that such nets can be defined analogously. We are going to prove that the spherical surfaces S whose intersections with a given spherical surface S^* are contained in diametral planes of the latter form a net. This will be done by showing that

$$(7) \qquad p(S, S^*) = p(S^*, S^*)$$

if S has the property in question. With usual notations, VIII.2(1) implies that (7) may be written

$$(8) \qquad \alpha\gamma^* + \gamma\alpha^* - b\bar{b}^* - \bar{b}b^* - 2\beta\frac{\alpha^*\gamma^* - b^*\bar{b}^*}{\beta^*} = 0$$

(in this form including planes S), and this in indeed a linear homogeneous relation between α, $\operatorname{Re} b$, $\operatorname{Im} b$, β, γ. In the proof we have to distinguish several cases.

Suppose that S^* is a sphere with center c^* and radius ϱ^*.

If S is a sphere, let c be its center, ϱ its radius, and $\sigma > 0$ the distance of c from c^*. That S intersects S^* in a great circle, amounts to that the triangle with c, c^*, and a common point of S and S^* as vertices has a right angle at c^*. Its opposite side has length ϱ and the other sides have lengths ϱ^* and σ. Hence, we have

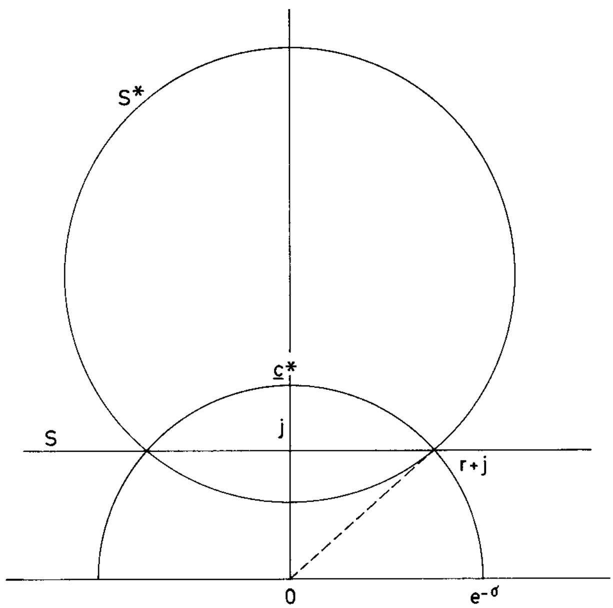
$$\cosh \varrho = \cosh \varrho^* \cosh \sigma$$

which may be written

$$\frac{\cosh \sigma}{\cosh \varrho \cosh \varrho^*} = \frac{1}{\cosh^2 \varrho^*},$$

and according to VIII.2(4), this is (7).

Let now S be a horosphere intersecting S^* in a great circle. It is convenient to assume it to be the horizontal e-plane through j, and also that the center c^* of S^* lies on the j-axis. The figure below shows the intersection of the configuration with the vertical e-plane through the real axis. Since VIII.2(7) will be applied, the



distance σ of c^* from S has to be provided with a sign, negative here since c^* lies in the interior of S. This implies that the center of S^* is $c^* = e^{-\sigma}j$, and that the e-radius of the diametral plane containing the intersection of S and S^* equals $e^{-\sigma}$. If $r + j$, $r > 0$, denotes the intersection point over the positive real axis, then

$$2r = 2(e^{-2\sigma} - 1)^{1/2}$$

is the length of the horocycle arc joining the intersection points $-r + j$ and $r + j$. Its chord has length $2\varrho^*$, and IV.5(2) therefore yields

$$e^{-2\sigma} - 1 = \sinh^2 \varrho^*$$

which is equivalent to

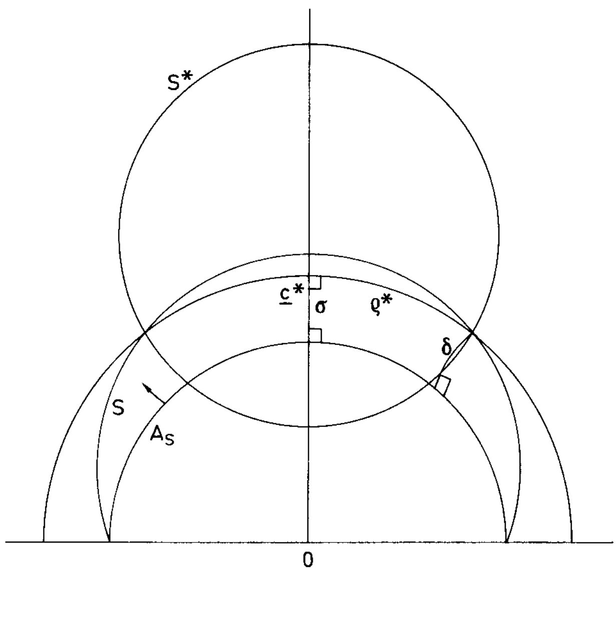
$$e^{-\sigma} = \cosh \varrho^* .$$

Hence,

$$\frac{e^{\sigma}}{\cosh \varrho^*} = \frac{1}{\cosh^2 \varrho^*}$$

and VIII.2(9) shows that (7) holds.

Let S now be an equidistant surface intersecting S^* in a great circle. The figure below shows the intersection with the vertical e-plane through the centre c* of S^* and the e-centre of the axial plane A_S of S. To simplify the figure both these



centres are assumed to lie on the j-axis. The distances δ of S from A_S and σ of c* from A_S have to be provided with signs according to an orientation of A_S. With the one indicated both are positive. The perpendiculars from c* and from one of the intersection points of S and S^* onto A_S have the lengths σ and δ, respectively. The intersection point, c* (whose distance from the latter equals, ϱ^*), and the foot points of the perpendiculars are the vertices of a quadrangle with three right

angles. Therefore we have

$$\sinh \delta = \cosh \varrho^* \sinh \sigma ,$$

and if we write this

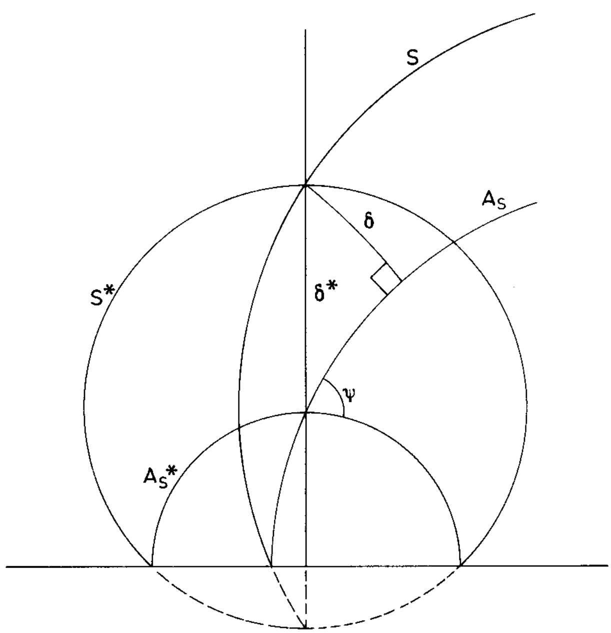
$$\frac{\sinh \sigma}{\sinh \delta \cosh \varrho^*} = \frac{1}{\cosh^2 \varrho^*} ,$$

VIII.2(5) shows that (7) also holds in this case.

The planes through the centre of S^* clearly belong to the net. They form an elliptic bundle of radical planes.

We now assume that the given spherical surface S^* is a horosphere. Let S be a spherical surface (necessarily an equidistant surface) whose intersection with S^* lies in a diametral plane of S^*. Since this implies that the horizon of S passes through the (improper) centre of S, we have $p(S, S^*) = 0$ by VIII.2(12), and $p(S^*, S^*) = 0$ by VIII.1(8). This proves (7) in this case. The planes whose horizons pass through the centre of S^* belong to the net and thus form a parabolic bundle of radical planes.

Finally we consider the case where S^* is an equidistant surface and S is a spherical surface (necessarily also an equidistant surface) whose intersection with S^* lies in a diametral plane of S^*. As usual, A_{S^*} and A_S denote the axial planes and δ^* and δ the distances of S^* and S from these. Since we shall use VIII.2(7), we have to choose the orientations of A_{S^*} and A_S, which here intersect, such that the angle ψ between the positive normals is acute or right. As mentioned at the end of



VIII.4, the plane containing $S^* \cap S$ contains also the line $A_{S^*} \cap A_S$. For the sake of simplicity we may assume this plane to be a vertical e-half-plane. The figure above shows the intersection of the configuration with the vertical e-plane through the e-centers of A_{S^*} and A_S. The perpendicular from the common point of S^* and S onto A_S has length δ. Its foot, the point of $S^* \cap S$, and the point of $A_{S^*} \cap A_S$ are the vertices of a right-angled triangle with the angle $\pi/2 - \psi$ opposite δ and the hypothenuse δ^*. Hence we have

$$\cos\psi = \frac{\sinh\delta}{\sinh\delta^*}.$$

If we write this

$$-\frac{\cos\psi}{\sinh\delta^*\sinh\delta} = \frac{1}{\sinh^2\delta^*},$$

VIII.2(7) and VIII.1(8) show that (7) also holds here. The diametral planes of S^*, that is, the planes orthogonal to A_{S^*}, belong to the net. They form a hyperbolic bundle of radical planes.

There seems to be no simple relation between the elliptic nets in the Euclidean sense and those in the hyperbolic sense defined above. In any case, the given e-sphere which is intersected in e-great circles is different from the e-sphere carrying the given spherical surface S^*.

Notes to Chapter VIII

A study of the spherical surfaces as in this chapter has, to the author's knowledge, not been carried out before. This holds also to a large extent for the study of circular curves in the hyperbolic plane. The specialization of the present results to the plane is left to the reader.

For the power of a point with respect to a circular curve in the hyperbolic plane see Sommerville [26] Chapt. VII, and Perron [19] §§ 39, 44, 51. Sommerville deals also with the radical axis of two circular curves.

In Section 5 dealing with linear families of spherical surfaces, there is referred to results concerning Euclidean linear families of e-spheres. For this topic see Coolidge [4] Chapt. V, § 2, and Sommerville [26] Chapt. 4, § 4, § 6.

It should be mentioned that investigations in which spherical surfaces play an essential role, and which are not included here, are due to Perron [19], [20], [21].

IX. Area and Volume

IX.1 Various coordinate systems

In this section we define several coordinate systems in hyperbolic space which are suitable for the calculation of areas and volumes. For this purpose we express the Riemannian metric and the volume element in terms of these coordinates.

Let M be a Riemannian manifold. For our purpose we may assume it to be homeomorphic with $\mathbb{R}^n (n = 2 \text{ or } 3)$. Let the metric be

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x^1, \ldots, x^n)\, dx^i dx^j$$

in terms of coordinates $x^1, \ldots, x^n$. Since the right-hand side is a positive definite quadratic form in $dx^1, \ldots, dx^n$ for every $(x^1, \ldots, x^n)$, we have

$$g = \det(g_{ij}) > 0\,.$$

The volume of a region $R \subset M$ is defined by

$$\operatorname{vol} R = \int \cdots \int_R \sqrt{g}\, dx^1 \ldots dx^n\,.$$

This is justified by the invariance of the integral under transformations of the coordinates and by the fact that it equals the volume in the case of a Euclidean space.

All the coordinate systems to be considered are orthogonal in the sense that coordinate curves, that is, curves on each of which only one coordinate varies, intersect orthogonally. This is equivalent to $g_{ij} = 0$ for $i \neq j$. Geometrically the latter means that the square of the infinitesimal displacement ds of a point p equals the sum of the squares of the displacements of p each due to the infinitesimal change dx^i of one of the coordinates. The "infinitesimal Pythagorean Theorem" will be used in a couple of cases to avoid tedious calculations.

Some coordinate systems do not satisfy $g > 0$ at the origin or an axis. Under calculations of areas and volume this can, however, be neglected as in the Euclidean case.

(a) So far only Euclidean orthonormal coordinate systems have been introduced. In the *upper half-space model* we used $\mathrm{x} = x + \xi j$, $x \in \mathbb{C}$, $\xi > 0$. For the Riemannian metric we had

$$ds^2 = \xi^{-2}(dx\, d\bar{x} + d\xi^2) = \xi^{-2}(dx_1^2 + dx_2^2 + d\xi^2)\,,$$

where $x = x_1 + x_2 i$, $x_1, x_2 \in \mathbb{R}$. Since $g = \xi^{-6}$, the volume element is

$$(1) \qquad d\,vol = \xi^{-3}\, dx_1\, dx_2\, d\xi\,.$$

For the plane $x_2 = 0$ we obtain

$$ds^2 = \xi^{-2}(dx_1^2 + d\xi^2)$$

and the area element

$$(2) \qquad d\,ar = \xi^{-2}\, dx_1\, d\xi\,.$$

If, instead of x_1, x_2 we introduce Euclidean polar coordinates r, φ by $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, we obtain for the Euclidean cylindric coordinates r, φ, ξ,

$$ds^2 = \xi^{-2}(dr^2 + r^2\, d\varphi^2 + d\xi^2)$$

and

$$(3) \qquad d\,vol = \xi^{-3} r\, dr\, d\varphi\, d\xi\,.$$

(b) In the *unit ball model* (cf. III.5) we had for the orthonormal e-coordinates

$$\begin{aligned} \mathrm{x}' &= x' + \xi' j, \quad x' \in \mathbb{C}, \quad \xi' \in \mathbb{R}\,, \quad x'\bar{x}' + \xi'^2 < 1\,, \\ ds^2 &= 4(1 - x'\bar{x}' - \xi'^2)^{-2}(dx'\, d\bar{x}' + d\xi'^2) \\ &= 4(1 - x_1'^2 - x_2'^2 - \xi'^2)^{-2}(dx_1'^2 + dx_2'^2 + d\xi'^2)\,, \end{aligned}$$

where $x' = x_1' + x_2' i$, $x'_1, x'_2 \in \mathbb{R}$. For the volume element this yields

$$(4) \qquad d\,vol = 8(1 - x_1'^2 - x_2'^2 - \xi'^2)^{-3}\, dx_1'\, dx_2'\, d\xi'\,.$$

For later use we define also Euclidean polar coordinates r', φ', ψ' in the unit ball as usual by

$$x_1' = r'\cos\varphi'\cos\psi'\,, \quad x_2' = r'\sin\varphi'\cos\psi'\,, \quad \xi' = r'\sin\psi'\,,$$

where $0 \le r' < 1$, $\varphi' \bmod 2\pi$, $-\pi/2 \le \psi' \le \pi/2$. Then we have as known from the Euclidean case

$$dx_1'^2 + dx_2'^2 + d\xi'^2 = dr'^2 + r'^2\cos^2\psi'\, d\varphi'^2 + r'^2\, d\psi'^2$$

and hence

$$(5) \qquad ds^2 = 4(1 - r'^2)^{-2}(dr'^2 + r'^2\cos^2\psi'\, d\varphi'^2 + r'^2\, d\psi'^2)\,.$$

The volume element therefore equals

$$(6) \qquad d\,vol = 8(1 - r'^2)^{-3}\, r'^2\cos\psi'\, dr'\, d\varphi'\, d\psi'\,.$$

We introduce now several types of coordinate systems defined in terms of notions and quantities of hyperbolic geometry. In the definitions we refer to an

arbitrarily chosen orthogonal frame, that is, an ordered triplet (A_1, A_2, A_3) of mutually orthogonal, oriented lines with a common point o, the origin (cf. III.2).

(c) *Horospherical coordinates.* Let S be the horosphere whose centre is the terminal point of A_3 and which passes through o. Then A_1 and A_2 are tangents of S at o. As observed in IV.5, the metric induced on S is Euclidean with the horocycles on S and concentric with S taking the roles of the e-lines. With those of these horocycles which touch A_1 and A_2 at o as coordinate axes we introduce rectangular e-coordinates y_1, y_2 on S. The horospherical coordinates y_1, y_2, η of a point p are then defined as follows: y_1, y_2 are the coordinates of the point of intersection of S and its diameter through p, and η is the distance of p from S measured on the diameter and positive or negative according as p is interior or exterior to S.

To determine the metric in terms of these coordinates we consider the upper half-space model. We may assume that S is the horizontal e-plane through j and that the y_1- and y_2-axes are situated such that $y_1 = x_1, y_2 = x_2$. As shown at the beginning of III.4, the distance from j to ξj equals $\log \xi$, and this is the coordinate η of p if its third e-coordinate equals ξ. With $\xi = e^\eta$ we obtain

$$ds^2 = e^{-2\eta} dy_1^2 + e^{-2\eta} dy_2^2 + d\eta^2$$

and thus

$$d\,vol = e^{-2\eta} dy_1\, dy_2\, d\eta\,, \tag{7}$$

and for the plane $y_2 = 0$ we have

$$\begin{aligned} ds^2 &= e^{-2\eta} dy_1^2 + d\eta^2\,, \\ d\,ar &= e^{-\eta} dy_1\, d\eta\,. \end{aligned} \tag{8}$$

(d) *Polar coordinates.* Choose a frame (A_1, A_2, A_3) with origin o. The polar coordinates ϱ, φ, ψ of a point p are defined as in e-space: $\varrho \geq 0$ is the distance of p from o, φ is the angle from A_1 to the projection op′ of the segment op onto the $A_1 A_2$-plane, and ψ the angle from op′ to op, both angles provided with signs.

To determine ds^2 in terms of these coordinates we use the unit-ball model. We may assume the frame (A_1, A_2, A_3) to coincide with the rectangular system used in (b). Then the angles φ and ψ are identical with φ' and ψ', respectively. However, here we have to use ϱ instead of the e-distance r'. Now according to III.5 we have

$$r' = \tanh \tfrac{1}{2}\varrho\,, \qquad \varrho = 2 \operatorname{Artanh} r'\,,$$

hence

$$d\varrho = 2(1 - r'^2)^{-1} dr'\,.$$

Since

$$\sinh \varrho = 2r'(1 - r'^2)^{-1}\,,$$

we obtain from (5)

$$ds^2 = d\varrho^2 + \sinh^2 \varrho \cos^2 \psi\, d\varphi^2 + \sinh^2 \varrho\, d\psi^2$$

and consequently

$$(9)\qquad d\,vol = \sinh^2\varrho \cos\psi \; d\varrho \; d\varphi \; d\psi \,.$$

For the plane $\psi = 0$ we have

$$\begin{aligned} ds^2 &= d\varrho^2 + \sinh^2\varrho \; d\varphi^2 \,, \\ (10)\qquad d\,ar &= \sinh\varrho \; d\varrho \; d\varphi \,. \end{aligned}$$

(e) *Cylindric coordinates.* The coordinates of a point p are now defined as follows: $\sigma \in \mathbb{R}$ is the coordinate on the axis A_3 of the orthogonal projection of p onto this axis, $\tau \geq 0$ is the distance of p from A_3, and $\varphi \bmod 2\pi$ is the polar coordinate used in (d).

To determine ds^2 in terms of these coordinates, we observe that if σ varies, while τ and φ are constant, p moves on an equidistant curve at distance τ from its axis A_3. As shown in IV.5, a change $d\sigma$ of σ causes a displacement of size $\cosh\tau \; d\sigma$ of p on the equidistant curve. If τ is changed by $d\tau$, while σ and φ are constant, p moves the same distance $d\tau$ along the normal to A_3 through p. If φ is changed by $d\varphi$, while σ and τ are constant, p moves on the circle with radius τ and centre at the foot of the perpendicular from p to A_3. Since this circle is isometric with an *e*-circle of *e*-radius $\sinh\tau$ (cf. IV.5), the length of the displacement of p is $\sinh\tau \; d\varphi$. Hence, we obtain

$$\begin{aligned} ds^2 &= \cosh^2\tau \; d\sigma^2 + d\tau^2 + \sinh^2\tau \; d\varphi^2 \,, \\ (11)\qquad d\,vol &= \cosh\tau \sinh\tau \; d\sigma \, d\tau \; d\varphi \,. \end{aligned}$$

For a constant value of φ this yields the metric in a half-plane in terms of σ and τ. To include the complementary half-plane we only have to provide τ with a sign in the obvious manner. The metric in a plane in terms of the "rectangular coordinates" σ, τ is therefore

$$(12)\qquad ds^2 = \cosh^2\tau \; d\sigma^2 + d\tau^2 \,.$$

(f) *Rectangular coordinates.* These coordinates η_1, η_2, η_3 of a point p are defined as follows. Let a frame (A_1, A_2, A_3) be chosen. Let p′ denote the orthogonal projection of p onto the A_1A_2-plane. Then η_1 is the A_1-coordinate of the orthogonal projection p″ of p′ onto A_1, η_2 is the distance from p″ to p′ provided with a sign according to the orientation of A_2, and η_3 is the distance from p′ to p provided with a sign according to the orientation of A_3.

In the plane $\eta_3 = 0$ the metric is

$$ds'^2 = \cosh^2\eta_2 \, d\eta_1^2 + d\eta_2^2$$

because (12) with η_1 and η_2 instead of σ and τ, respectively, can be applied. Let E denote the equidistant surface through p with axial plane $\eta_3 = 0$. An infinitesimal displacement ds' of p′ causes a displacement of p on E which, as shown in IV.5,

has the size $\cosh\eta_3\, ds'$ since $|\eta_3|$ is the distance of E from its axial plane. An infinitesimal change $d\eta_3$ of η_3 causes a displacement of p of the same size in the direction orthogonal to E. For the total line element ds we therefore have

$$\begin{aligned} ds^2 &= \cosh^2\eta_3\, ds'^2 + d\eta_3^2 \\ &= \cosh^2\eta_2 \cosh^2\eta_3\, d\eta_1^2 + \cosh^2\eta_3\, d\eta_2^2 + d\eta_3^2 \end{aligned}$$

and hence

$$d\,vol = \cosh\eta_2 \cosh^2\eta_3\, d\eta_1\, d\eta_2\, d\eta_3\,. \tag{13}$$

For the plane $\eta_3 = 0$ we have

$$d\,ar = \cosh\eta_2\, d\eta_1\, d\eta_2\,. \tag{14}$$

IX.2 Area

We start by observing that some previous results yield information about the area of regions on certain surfaces.

As shown in III.5, a sphere of radius ϱ is isometric with an e-sphere of e-radius $\sinh\varrho$. Hence, the area of a region on the sphere equals the e-area of its image on the e-sphere. In particular, the *surface area of the sphere* equals $4\pi\sinh^2\varrho$.

In IV.5 it was observed that a *horosphere* is isometric with an e-plane. This reduces the determination of the area to the Euclidean case. The same holds for an *equidistant cylinder* of radius ϱ, which has been shown to be isometric with an e-cylinder of e-radius $\sinh\varrho$ (cf. IV.5).

We consider now a plane H. We may assume it to be the vertical e-half-plane bounded by the real axis. By 1.(2) the area of a region $R \subset H$ equals

$$ar\, R = \int_R \xi^{-2}\, dx_1\, d\xi\,.$$

Let R be a right-angled triangle with an improper vertex. We may assume the latter to be ∞ and the vertex of the right angle to be j. Then two of the sides are vertical e-half-lines and the third is an arc of the e-circle with e-center 0 and e-radius 1 joining j with the third vertex. If α denotes the angle at this vertex, the latter is $\cos\alpha + j\sin\alpha$ (provided the triangle lies to the "right" of the j-axis). Clearly, $0 \le \alpha < \pi/2$. For the area of the triangle we have

$$\begin{aligned} \int_0^{\cos\alpha} \int_{(1-x_1^2)^{1/2}}^{\infty} \xi^{-2}\, dx_1\, d\xi &= \int_0^{\cos\alpha} (1 - x_1^2)^{-1/2}\, dx_1 \\ &= -\Big[\operatorname{Arc\,cos} x_1\Big]_0^{\cos\alpha} = \pi/2 - \alpha\,. \end{aligned}$$

If $\alpha = 0$, a second vertex is improper.

The area of a triangle R with an improper vertex and non-right angles α and β at the other vertices is the sum or the difference of the areas of two right-angled triangles with the same improper vertex. Indeed, the altitude from the improper vertex is either in the interior of the triangle, then α and β are acute, and $ar\,R = \pi/2 - \alpha + \pi/2 - \beta$, or it lies outside, then one of the angles, α say, is obtuse, and $ar\,R = \pi/2 - \beta - (\pi/2 - (\pi - \alpha))$. Hence, in both cases

$$ar\,R = \pi - \alpha - \beta\,.$$

This holds also if $\alpha = 0$ or/and $\beta = 0$.

To find the area of a *triangle* R with proper vertices A, B, C (we use here the customary notation) and angles α, β, γ, draw the half-line from A through B and the half-line from C parallel to it. Denote the common end of these half-lines by D and the angle BCD by φ. The area of R is the difference of the areas of the triangles ADC and BCD, thus

$$\begin{aligned} ar\,R &= \pi - \alpha - (\gamma + \varphi) - (\pi - (\pi - \beta) - \varphi) \\ &= \pi - \alpha - \beta - \gamma\,. \end{aligned}$$

According to the previous results this holds also if the triangle has improper vertices. It shows again (cf. VI.3) that $\pi - \alpha - \beta - \gamma > 0$. This quantity is called the *defect* of the triangle.

Let R now be a *convex polygon* with m, possibly improper, vertices, and let its interior angles be φ_ν, $\nu = 1, \ldots, m$. The segments joining an interior point with the vertices divide R into m triangles. The sum of their angles equals $2\pi + \sum_{\nu=1}^{m} \varphi_\nu$. Consequently,

$$(1) \qquad ar\,R = (m-2)\pi - \sum_{\nu=1}^{m} \varphi_\nu\,.$$

More generally we consider a *polygonal region* R, that is, an open subset of the plane bounded by finitely many simple closed polygons. Improper vertices are again admitted. It is not excluded that the polygons have isolated points in common, while their interiors are of course disjoint. Let n denote the total number of distinct vertices of the bounding polygons and ϕ the sum of all interior angles of R. The extension of the sides of the bounding polygons divide R into finitely many convex polygons. Let ε_2 denote their number, ε_1 the number of distinct sides, and ε_0 the number of distinct vertices of the subdivision. Further, let ε_1' and ε_0' be the numbers of those of these sides and vertices, respectively, which belong to the boundary of R. The numbers

$$\chi = \varepsilon_0 - \varepsilon_1 + \varepsilon_2$$
and
$$\chi' = \varepsilon_0' - \varepsilon_1'$$

are the *Euler characteristic* and the *boundary characteristic* of the subdivision. We

claim that

$$(2) \qquad ar\,R = (m - 2\chi + \chi')\pi - \phi\,.$$

To prove it we apply (1) to the polygons of the subdivision. Addition of the terms $m\pi$ yields $(2\varepsilon_1 - \varepsilon_1')\pi$ since the sides in the interior of R are counted twice and those on the boundary once. Addition of the terms 2π yields $2\varepsilon_2\pi$. The sum of the angle sums is ϕ plus a contribution 2π for each of the vertices, in the interior of R, thus $2(\varepsilon_0 - \varepsilon_0')\pi$, further a contribution π for each of those vertices which are inner points of sides of the original boundary, thus $(\varepsilon_0' - n)\pi$. Hence, the area of R equals

$$(2\varepsilon_1 - \varepsilon_1')\pi - 2\varepsilon_2\pi - \phi - 2(\varepsilon_0 - \varepsilon_0')\pi - (\varepsilon_0' - n)\pi$$

and this is the statement (2).

It is obvious that the boundary characteristic χ' is independent of the subdivision. Clearly $\chi' = 0$ if the polygons bounding R are mutually disjoint. The well-known fact that χ is independent of the subdivision follows here for subdivision into convex polygons.

We determine the area of some other simple regions in a plane. By 1.(10) we have for the area of a *disc* R with radius ϱ_0

$$\begin{aligned} ar\,R &= \int_0^{\varrho_0}\int_0^{2\pi} \sinh\varrho\, d\varrho\, d\varphi \\ &= 2\pi(\cosh\varrho_0 - 1) = 4\pi\sinh^2\frac{\varrho_0}{2}\,. \end{aligned}$$

Let R be the *region bounded by two concentric horocycles and by two diameters*. Denote by l the length of the bounding arc of the outer horocycle and by $\delta > 0$ the distance from the outer to the inner horocycle. We use horocyclic coordinates y_1, η chosen such that the outer horocycle has equation $\eta = 0$, the inner one equation $\eta = \delta$, and the bounding diameters equations $y_1 = 0$ and $y_1 = l$. By 1.(8) we then have

$$ar\,R = \int_0^{l}\int_0^{\delta} e^{-\eta}\, dy_1\, d\eta = l(1 - e^{-\delta})\,.$$

Here the value $\delta = +\infty$ is permitted, so we obtain for the area of a *horocycle sector*, whose bounding horocycle arc has length l,

$$ar\,R = l\,.$$

Let R be the *region bounded by a horocycle and a line* which intersect at two distinct points. Denote by ω the angle of intersection and by l the length of the

bounding horocycle arc. The area of R is the difference of the area l of the horocycle sector bounded by the diameters through the points of intersection and the area of an isosceles triangle with one improper vertex and angles of size $\pi/2 - \omega$ at the other vertices. Hence,

$$ar\, R = l - 2\,\omega\,.$$

There is a simple relation between l and ω. To obtain it, assume the horocycle to be the e-line $x_2 = 0$, $\xi = 1$, and the midpoint of the horocycle arc to be j. In the right-angled e-triangle with vertices $0, j, j + l/2$ the angle at 0 *equals* ω. We therefore have $l/2 = \tan\omega$, and thus

$$ar\, R = 2\,(\tan\omega - \omega) = l - 2 \operatorname{Arctan} l/2\,.$$

Finally let R be the *region bounded by an equidistant curve, its axis, and two diameters*. Denote by δ the distance of the equidistant curve from its axis and by λ the length of the segment of the axis which joins the bounding diameters. We use rectangular coordinates with A_1 coinciding with the axis and A_2 with one of the two diameters. By 1.(14) we then have, with suitable orientations of A_1 and A_2,

$$ar\, R = \int_0^{\lambda}\int_0^{\delta} \cosh\eta_2\, d\eta_1\, d\eta_2 = \lambda \sinh\delta\,.$$

We add the remark that determination of areas on an *equidistant surface* reduces to that in a plane. Indeed, as shown in IV.5(4): under the bijection of the axial plane onto the surface by means of the diameters the line element is multiplied by $\cosh\delta$, where δ denotes the distance. This implies that the area of a region on the surface is $\cosh^2\delta$ times the area of the corresponding region in the axial plane.

IX.3 Volume of some bodies of revolution

For a *sphere* R of radius ϱ_0 we obtain by 1.(9)

$$\begin{aligned} vol\, R &= \int_0^{\varrho_0}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \sinh^2\varrho \cos\psi\, d\varrho\, d\varphi\, d\psi \\ &= 4\pi \int_0^{\varrho_0} \sinh^2\varrho\, d\varrho = 2\pi \int_0^{\varrho_0} (\cosh 2\varrho - 1)\, d\varrho \\ &= \pi(\sinh 2\varrho_0 - 2\varrho_0)\,. \end{aligned}$$

An *equidistant cylinder* R is a region which is bounded by the equidistant curves with a common axis and with the same distance δ from it and by two discs with centers on the axis and contained in planes orthogonal to it. Let κ denote the altitude, that is, the distance of the discs. With A_1, A_2 in the plane of one of the discs and A_3 coinciding with the axis of the cylinder, we obtain by 1.(11), using cylindric coordinates,

$$vol\, R = \int_0^{\kappa}\int_0^{\delta}\int_0^{2\pi} \sinh\tau \cosh\tau \; d\sigma\, d\tau\, d\varphi = \pi\kappa \sinh^2\delta .$$

Let R now denote a *horosphere cylinder*, that is, a region bounded by a region R_0 on a horosphere S, the half-lines from the boundary points of R_0 to the center of S, and a concentric horosphere S' in the interior of S. Here horospherical coordinates are appropriate. Choose the y_1- and y_2-axes on S and the η-axis oriented towards the interior of S. If δ denotes the distance from S to S', we then have by 1.(7)

$$vol\, R = \iint_{R_0}\int_0^{\delta} e^{-2\eta}\, dy_1\, dy_2\, d\eta = \tfrac{1}{2}(1 - e^{-2\delta})\, ar\, R_0 .$$

This makes also sense for $\delta = +\infty$ and then yields the volume

$$vol\, R = \tfrac{1}{2}\, ar\, R_0$$

of the *horosphere sector* bounded by the region R_0 on S and the diameters of S from the boundary points of R_0 to the center of S.

Next we consider a *cone of revolution with improper vertex*, that is, the region R obtained by rotating a right-angled triangle with one improper vertex about the side opposite the proper vertex with acute angle α. Here the Euclidean cylindric coordinates r, φ, ξ in the upper half-space are most convenient. We may suppose that the axis of the cone is the j-axis and that the plane containing the base disc is the e-hemisphere, with center 0 and e-radius 1. The meridians are then vertical e-half-lines at e-distance $\cos\alpha$ from the j-axis. If ϱ_0 denotes the radius of the base disc, we have

$$\sin\alpha \cosh\varrho_0 = 1$$

since α is the angle of parallelisme for the distance ϱ_0 (cf. VI.3.5). By 1.(3) the volume of the cone equals

$$\int_0^{\cos\alpha}\int_0^{2\pi}\int_{\sqrt{1-r^2}}^{\infty} r\xi^{-3}\, dr\, d\varphi\, d\xi = \pi \int_0^{\cos\alpha} r(1-r^2)^{-1}\, dr = \tfrac{1}{2}\pi\Big[\log(1-r^2)\Big]_{\cos\alpha}^{0} ,$$

hence

$$vol\,R = -\pi \log \sin \alpha = \pi \log \cosh \varrho_0 .$$

Let R now denote the *region bounded by a horosphere and an intersecting plane*. Let ω denote the acute angle of intersection and ϱ_0 the radius of the flat bounding disc. The (horocyclic) radius of the bounding horosphere disc then equals $\sinh \varrho_0$ (cf. IV.5(1)). The area of this disc is $\pi \sinh^2 \varrho_0$. Now the volume of R equals the volume of the horosphere sector bounded by the horosphere disc minus the volume of the cone of revolution with improper vertex and radius ϱ_0 of the basic flat disc. Hence, by the results above

$$vol\,R = \tfrac{1}{2}\pi \sinh^2 \varrho_0 - \pi \log \cosh \varrho_0 .$$

To express the volume in terms of ω we observe that the angle α used above equals $\frac{1}{2}\pi - \omega$, so $\cosh \varrho_0 = 1/\cos \omega$ and consequently $\sinh \varrho_0 = \tan \omega$. Hence,

$$vol\,R = \tfrac{1}{2}\pi \tan^2 \omega + \pi \log \cos \omega .$$

Next we consider a *region R bounded by an equidistant surface, its axial plane, and diameters of it*. Let $\delta > 0$ be the distance from the axial plane and R_0 the region on the latter which belongs to the boundary of R. Here we use rectangular coordinates η_1, η_2, η_3. We choose a point of the axial plane as the origin o of the frame (A_1, A_2, A_3) and as A_3 the diameter through o oriented towards the equidistant surface. Using 1.(13) we obtain for the volume of R

$$\int\limits_{R_0}\!\!\int \int\limits_0^{\delta} \cosh \eta_2 \cosh^2 \eta_3 \, d\eta_1 \, d\eta_2 \, d\eta_3$$

$$= \tfrac{1}{4}(\sinh 2\delta + 2\delta) \int\limits_{R_0}\!\!\int \cosh \eta_2 \, d\eta_1 \, d\eta_2$$

and thus by 1.(14)

$$\mathrm{vol}\, R = \tfrac{1}{4}(\sinh 2\delta + 2\delta)\, ar\, R_0 .$$

Let R now be a *cone of revolution*. Let ϱ_0 denote the radius of the planar base disc, κ the altitude, μ the length of a meridian, and α the angle between the axis and a meridian. Since these quantities are the sides and an angle of a right-angled triangle, we have (cf. VI.3(5))

$$\cos \alpha = \tanh \kappa \coth \mu , \quad \sinh \kappa = \tanh \varrho_0 \cot \alpha .$$

To compute the volume we use cylindric coordinates σ, τ, φ. We choose the vertex of the cone as the origin and its axis as the axis A_3. For the points p in the cone φ takes all values mod 2π and σ varies from 0 to κ. The interval $[\![0, \tau_\sigma]\!]$

through which τ runs depends on σ. For the length τ_σ of the segment through p orthogonal to A_3 and leading from A_3 to a meridian we obtain by considering a right-angled triangle

$$\tanh \tau_\sigma = \tanh \alpha \sinh \sigma .$$

According to 1.(11) the volume of the cone equals

$$\begin{aligned}
vol\, R &= \int_0^\kappa \int_0^{\tau_\sigma} \int_0^{2\pi} \cosh \tau \sinh \tau \, d\sigma \, d\tau \, d\varphi \\
&= \pi \int_0^\kappa (\cosh^2 \tau_\sigma - 1) \, d\sigma \\
&= \pi \int_0^\kappa \frac{1}{1 - \tan^2 \alpha \sinh^2 \sigma} \, d\sigma - \pi\kappa \\
&= \pi \cos \alpha \int_0^\kappa \frac{1}{1 - \tanh^2 \sigma / \cos^2 \alpha} \cdot \frac{1}{\cosh^2 \sigma \cos \alpha} \, d\sigma - \pi\kappa \\
&= \pi \cos \alpha \operatorname{Artanh}(\tanh \kappa / \cos \alpha) - \pi\kappa \\
&= \pi(\mu \cos \alpha - \kappa) .
\end{aligned}$$

By means of the relations above the volume can be expressed in terms of any two of ϱ_0, κ, μ, α.

Finally we let R denote a *rectilinear cylinder of revolution*. It is obtained by rotating a quadrangle with three right angles about one of the sides joining the vertices of two right angles. Let κ denote the length of this side, further ϱ_0 the length of the other side joining the vertices of two right angles, and μ the length of the side opposite to that of length κ. Specializing a relation for a quadrangle with two right angles (cf. VI.3.3), say) yields

$$\cosh \varrho_0 = \tanh \mu \coth \kappa .$$

Clearly, κ is the altitude of the cylinder, ϱ_0 the radius of the base disc, and μ the length of the meridian.

We use again cylindric coordinates σ, τ, φ and choose as the origin the center of the base disc and as A_3 the axis of the cylinder. For the points p in the cylinder φ takes all values mod 2π and σ varies from 0 to κ. The length τ_σ of the interval through which τ runs for a given σ equals also here the length of the segment through p orthogonal to the axis and joining the latter and a meridian. Consider-

ing that ϱ_0, σ, τ_σ are the lengths of neighbouring sides of a quadrangle with three right angles, we see that

$$\tanh \tau_\sigma = \cosh \sigma \tanh \varrho_0 .$$

For the volume of the cylinder we therefore have

$$\begin{aligned}
\operatorname{vol} R &= \int_0^{\kappa} \int_0^{\tau_\sigma} \int_0^{2\pi} \cosh \tau \sinh \tau \, d\sigma \, d\tau \, d\varphi \\
&= \pi \int_0^{\kappa} (\cosh^2 \tau_\sigma - 1) \, d\sigma \\
&= \pi \int_0^{\kappa} \frac{1}{1 - \cosh^2 \sigma \tanh^2 \varrho_0} \, d\sigma - \pi\kappa \\
&= \pi \int_0^{\kappa} \frac{1}{1 - \tanh^2 \sigma \cosh^2 \varrho_0} \cdot \frac{\cosh^2 \varrho_0}{\cosh^2 \sigma} \, d\sigma - \pi\kappa \\
&= \pi \cosh \varrho_0 \operatorname{Artanh} (\tanh \kappa \cosh \varrho_0) - \pi\kappa \\
&= \pi (\mu \cosh \varrho_0 - \kappa) .
\end{aligned}$$

By means of the relation above between ϱ_0, κ, μ the volume can be expressed in terms of any two of these quantities.

IX.4 Volume of polyhedra

As in Euclidean space the determination of the volume of a polyhedron can be reduced to that of tetrahedra. A convex polyhedron can obviously be divided into tetrahedra, and a non-convex one is divided into convex ones by the planes containing its faces. However, in hyperbolic space even the determination of the volume of a tetrahedron is very involved. Therefore one considers a special type of tetrahedra, the so-called *orthoschemes*, with the property that the dihedral angles at three non-coplanar edges are right. As will be shown, every tetrahedron can be divided into at most six orthoschemes.

It is convenient here to use the customary notations: capital latin letters for points, lower case latin latters for lengths of segments, and lower case greek letters for angles. In particular, the rectangular coordinates will now be denoted by x, y, z.

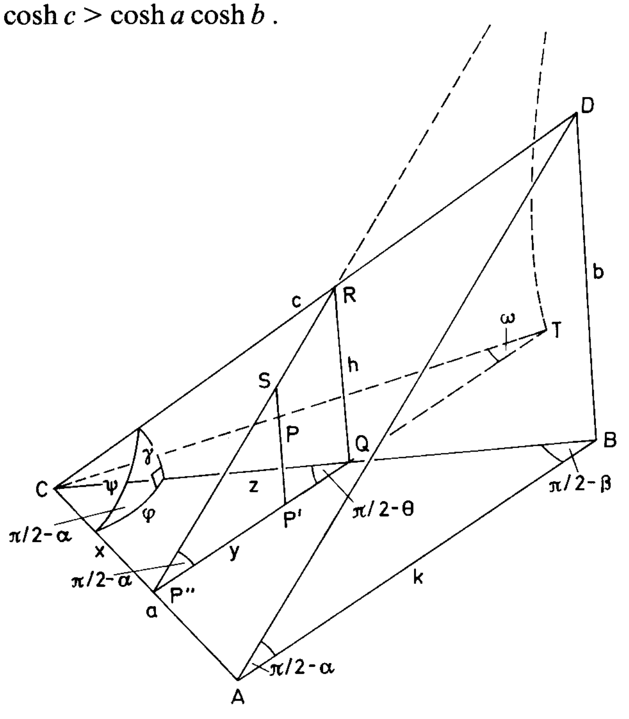
Let A, B, C, D be the vertices of an arbitrary tetrahedron, E the foot of the perpendicular from D to the plane ABC, and A', B', C' the feet of the perpendiculars from E to the lines BC, CA, AB, respectively. The six tetrahedra $AC'ED$, $C'BED$, $BA'ED$, $A'CED$, $CB'ED$, $B'AED$ are orthoschemes since the dihedral angles of the first one at the edges AE, EC', $C'D$ are right, and analogously for the others. If all of the feet E, A', B', C' belong to the tetrahedron $ABCD$, the latter is divided into the six orthoschemes. If this is not the case, some of the latter have to be counted negatively to obtain $ABCD$.

From now on we consider an orthoscheme $ABCD$ whose dihedral angles at the edges DA, AB, BC are right. This implies that the edge CA is orthogonal to the plane ABD, in particular to the edge AB, and that the edge BD is orthogonal to the plane ABC, in particular to the edge AB. Clearly, the segments CA, AB, BD with this mutual position determine the orthoscheme. Their lengths a, k, b can be prescribed arbitrarily. It is convenient to use the length c of CD instead of the length k of AB. However, a, b, c cannot be given arbitrarily. The right-angled triangles ACD and BCD show that $c > a$ and $c > b$, but this is not sufficient. The relations between the sides of these triangles yield

$$\cosh k = \frac{\cosh c}{\cosh a \cosh b}.$$

Hence, a, b, c determine an orthoscheme if and only if

$$\cosh c > \cosh a \cosh b.$$



In the determination of the volume the non-right dihedral angles at the edges AC, BD, CD play a decisive role. For convenience they are denoted by $\pi/2-\alpha$, $\pi/2-\beta, \gamma$, respectively. We notice that any relation between $a, b, c, \alpha, \beta, \gamma$ remains valid if a, α are interchanged with b, β. In the sequel this will be used tacitely. To derive relations of this kind we use the spherical triangle obtained by intersecting the orthoscheme with a sphere with center C. Its angle at the edge CB is right, that at CA equals $\pi/2-\alpha$, and that at CD equals γ. Since $\pi/2-\alpha+\gamma>\pi/2$, we have

$$(1) \qquad \alpha<\gamma\,, \qquad \beta<\gamma\,.$$

Let φ denote the side opposite γ, thus $\angle ACB$, and ψ the hypothenuse, thus $\angle ACD$. Among the relations between these angles and sides we shall use

$$(2) \qquad \cos\gamma=\cos\alpha\cos\varphi\,,$$

$$(3) \qquad \tan\alpha=\tan\gamma\cos\psi\,.$$

Since φ and $\pi/2-\beta$ are the acute angles of the right-angled triangle ABC, we have $\varphi<\beta$ and hence

$$(4) \qquad \cos\gamma>\cos\alpha\cos\beta\,.$$

In this triangle we further have

$$\sin\beta=\cosh a\sin\varphi\,,$$

so by (2)

$$(5) \qquad \cosh^2 a=\frac{\cos^2\alpha\sin^2\beta}{\cos^2\alpha-\cos^2\gamma}\,, \quad \cosh^2 b=\frac{\cos^2\beta\sin^2\alpha}{\cos^2\beta-\cos^2\gamma}$$

It is easily seen that (1) and (4) imply that the right-hand sides are greater than one, so there are $a>0$ and $b>0$ satisfying (5). Considering again the triangle ABC we see that

$$\tanh k=\sinh a\tan\varphi\,,$$

thus, by (5) and (2),

$$\tanh^2 k=\left(\frac{\cos^2\alpha\sin^2\beta}{\cos^2\alpha-\cos^2\gamma}-1\right)\left(\frac{1-\cos^2\gamma/\cos^2\alpha}{\cos^2\gamma/\cos^2\alpha}\right)$$

$$=\frac{\cos^2\alpha\sin^2\beta-\cos^2\alpha+\cos^2\gamma}{\cos^2\gamma}=\frac{\cos^2\gamma-\cos^2\alpha\cos^2\beta}{\cos^2\gamma}$$

and the right-hand side is positive by (4) and less than 1, so a unique $k>0$ satisfying this exists. Together these results show that α, β, γ determine an orthoscheme, and only one, if and only if they satisfy (1) and (4).

We turn now to the determination of the volume of the orthoscheme. To introduce rectangular coordinates we choose C as the origin of the frame (A_1, A_2, A_3), the line AC as the axis A_1, and A_2 in the plane ABC. Let the orientations be chosen such that points of the orthoscheme have positive coordinates. If P is such a point, let P' be the orthogonal projection of P onto the plane ABC and P'' the orthogonal projection of P' onto CA. Then

$$x = PP'' , \quad y = P''P' , \quad z = P'P .$$

Let Q and R denote the points at which the plane $PP'P''$ meets the edges BC and CD, respectively, and S the point of intersection of the lines $P'P$ and $P''R$. We introduce the notations

$$P''Q = f(x) , \qquad P'P = g(y) ,$$

so

$$QR = g(f(x)) .$$

The length of $P'P$ depends indeed only on y since the plane $PP'P''$ is orthogonal to AC. Considering the right-angled triangles $CP''Q$ and $SP''P'$ one obtains, using (2),

$$\tan f(x) = \sinh x \tan \varphi = \frac{\sinh x}{\cos \gamma} \sqrt{\cos^2 \alpha - \cos^2 \gamma} , \tag{6}$$

$$\tanh g(y) = \sinh y \cot \alpha . \tag{7}$$

For the points P of the orthoscheme x varies from 0 to a. For x fixed, y varies from 0 to $f(x)$, and for y fixed, z varies from 0 to $g(y)$. Hence, by 1.(12) the volume equals

$$\begin{aligned} vol &= \int_0^a \int_0^{f(x)} \int_0^{g(y)} \cosh y \cosh^2 z \, dx \, dy \, dz \\ &= \frac{1}{4} \int_0^a \int_0^{f(x)} \cosh y \, (\sinh 2g(y) + 2g(y)) \, dx \, dy \\ &= \frac{1}{4} \int_0^a \Big[2g(y) \sinh y \Big]_0^{f(x)} dx \\ &= \frac{1}{2} \int_0^a g(f(x)) \sinh f(x) \, dx . \end{aligned}$$

Here is used that, by (7),

$$2\sinh y\,\frac{dg(y)}{dy} = 2\cosh y\sinh y\cot\alpha\cosh^2 g(y)$$
$$= 2\cosh y\tanh g(y)\cosh^2 g(y)$$
$$= \cosh y\sinh 2g(y)\,.$$

If one inserts the expressions for f and g following from (6) and (7) in the last integral, and uses the expression for a following from (5), one obtains the volume in terms of α, β, γ. However, the integrand is very complicated.

The integral can be simplified by introducing a new variable, namely the angle CQP'' of the above figure which we shall denote by $\pi/2 - \vartheta$. Considering the right-angled triangle $CP''Q$, we see that

$$\sin\vartheta = \sin\varphi\cosh x$$

(8) $$\sinh(f(x) = \tan\vartheta\tanh x$$

which imply

$$\sin\varphi\sinh x = \cos\vartheta\sinh f(x)\,.$$

Hence we have

$$\cos\vartheta\, d\vartheta = \sin\varphi\sinh x\, dx = \cos\vartheta\sinh f(x)\, dx\,.$$

If x varies from 0 to a, ϑ varies from φ to β. Setting

$$g(f(x)) = h(\vartheta)\,,$$

the expression above for the volume can be written

(9) $$vol = \frac{1}{2}\int_{\varphi}^{\beta} h(\vartheta)\, d\vartheta$$

which admits of a surprisingly simple geometric interpretation.

To make the dependence of the integral of α, β, γ explicit, we observe that the triangle $CP''Q$ and (2) yield

$$\cosh f(x)\frac{\cos\varphi}{\cos\vartheta} = \frac{\cos\gamma}{\cos\alpha\cos\vartheta}\,.$$

Hence we obtain from (7)

(10) $$\tanh g(f(x)) = \cot\alpha\sinh f(x)$$
$$= \frac{\sqrt{\cos^2\gamma - \cos^2\alpha\cos^2\beta}}{\sin\alpha\cos\vartheta} = \tanh h(\vartheta)\,,$$

thus

$$(11)\qquad vol = \frac{1}{2}\int_{\varphi}^{\beta} \operatorname{Artanh} \frac{\sqrt{\cos^2\gamma - \cos^2\alpha\cos^2\beta}}{\sin\alpha\cos\vartheta}\, d\vartheta$$

where, by (2),

$$\varphi = \operatorname{Arc\,cos} \frac{\cos\gamma}{\cos\alpha}.$$

Finally we are going to derive a quite different expression for the volume, namely as a sum and difference of values of a (non-elementary) function of one variable, the *Lobatčefskii function*

$$(12)\qquad L(\xi) = -\int_0^{\xi} \log(2\cos\eta)\, d\eta\,, \quad \xi \in [\![-\pi/2, \pi/2]\!]\,.$$

Clearly, $L(-\xi) = -L(\xi)$. We note further that $L(\pi/2) = 0$. To see this, observe that also

$$-\int_0^{\pi/2} \log(2\sin\eta)\, d\eta = L(\pi/2)$$

and hence

$$\begin{aligned} 2L(\pi/2) &= -\int_0^{\pi/2} (\log(2\cos\eta) + \log(2\sin\eta))\, d\eta \\ &= -\int_0^{\pi/2} \log(2\sin 2\eta)\, d\eta \\ &= -\frac{1}{2}\int_0^{\pi} \log(2\sin\zeta)\, d\zeta = L(\pi/2)\,. \end{aligned}$$

For the purpose in question it is necessary to introduce the angle $\omega(\vartheta)$ defined as follows. Let T denote the foot of the perpendicular from the end of the half-line from P'' through R on the line $P''Q$ (clearly also on the plane ABC). Then

$$\omega(\vartheta) = \angle P''TC\,.$$

Since $P''T$ is the distance whose parallel-angle is $\pi/2 - \alpha$, so $\sinh(P''T) = \tan\alpha$, we obtain, considering the right-angled triangle $P''TC$,

$$(13)\qquad \tan\omega(\vartheta) = \frac{\tanh x}{\tan\alpha}\,.$$

Now by (8) and (2)

$$\tanh^2 x = 1 - \frac{1}{\cosh^2 x} = 1 - \frac{\sin^2 \varphi}{\sin^2 \vartheta}$$
$$= \frac{\cos^2 \varphi - \cos^2 \vartheta}{\sin^2 \vartheta} = \frac{\cos^2 \gamma - \cos^2 \alpha \cos^2 \vartheta}{\cos^2 \alpha \sin^2 \vartheta},$$

hence

$$\tan^2 \omega(\vartheta) = \frac{\cos^2 \gamma - \cos^2 \alpha \cos^2 \vartheta}{\sin^2 \alpha \sin^2 \vartheta}. \tag{14}$$

This relation is needed in the very different form

$$\begin{aligned}\cos(\alpha+\omega)\cos(\alpha-\omega)\cos(\vartheta+\omega)\cos(\vartheta-\omega)& \\ - \cos(\gamma+\omega)\cos(\gamma-\omega)\cos^2\omega &= 0.\end{aligned} \tag{15}$$

Indeed, after division by $\cos^4 \omega$ the left-hand side may be written

$$\begin{aligned}&(\cos^2\alpha - \sin^2\alpha\tan^2\omega)(\cos^2\vartheta - \sin^2\vartheta + \tan^2\omega) \\ &\qquad - (\cos^2\gamma - \sin^2\gamma\tan^2\omega) \\ &= \cos^2\alpha\cos^2\vartheta - \cos^2\gamma + \sin^2\alpha\sin^2\vartheta\tan^4\omega \\ &\qquad - (\cos^2\alpha\sin^2\vartheta + \sin^2\alpha\cos^2\vartheta - \sin^2\gamma)\tan^2\omega \\ &= (-\sin^2\alpha\sin^2\vartheta + \cos^2\gamma - \cos^2\alpha\cos^2\vartheta \\ &\qquad - \cos^2\alpha\sin^2\vartheta + \sin^2\gamma - \sin^2\alpha\cos^2\vartheta)\tan^2\omega = 0.\end{aligned}$$

Here (14) is used twice.

Further we use (14) to write the expression (10) for $h(\vartheta)$ in a more suitable form:

$$\begin{aligned}h(\vartheta) &= \operatorname{Artanh}(\tan\vartheta\tan\omega(\vartheta)) \\ &= \frac{1}{2}\log\frac{1+\tan\vartheta\tan\omega(\vartheta)}{1-\tan\vartheta\tan\omega(\vartheta)},\end{aligned}$$

hence

(16 $$h(\vartheta) = \frac{1}{2}\log\frac{\cos(\vartheta-\omega(\vartheta))}{\cos\vartheta+\omega(\vartheta))}.$$

The statement to be proved is

$$\begin{aligned}4vol = L(\alpha+\delta) - L(\alpha-\delta) + L(\beta+\delta) - L(\beta-\delta)& \\ - L(\gamma+\delta) + L(\gamma-\delta) - 2L(\delta)&,\end{aligned} \tag{17}$$

where

$$\delta = \omega(\beta).$$

For this purpose we consider the function

$$\phi(\vartheta,\omega) = L(\alpha+\omega) - L(\alpha-\omega) + L(\vartheta+\omega) - L(\vartheta-\omega) \\ - L(\gamma+\omega) + L(\gamma-\omega) - 2L(\omega)\,.$$

We have to show that $\phi(\varphi,\omega(\varphi)) = 0$ and that

$$\frac{d\phi(\vartheta,\omega(\vartheta))}{d\vartheta} = 2h(\vartheta)\,.$$

For $\vartheta = \varphi$ we have $x = 0$, so (13) implies $\omega(\varphi) = 0$ and hence $\phi(\varphi,\omega(\varphi)) = 0$. To prove the second statement we use that

$$\frac{d\phi(\vartheta,\omega(\vartheta))}{d\vartheta} = \frac{\partial\phi}{\partial\vartheta} + \frac{\partial\phi}{\partial\omega}\frac{d\omega}{d\vartheta}\,.$$

Now

$$\begin{aligned}\frac{\partial\phi}{\partial\omega} = &-\log(2\cos(\alpha+\omega)) - \log(2\cos(\alpha-\omega)) \\ &-\log(2\cos(\vartheta+\omega)) - \log(2\cos(\vartheta-\omega)) \\ &+\log(2\cos(\gamma+\omega)) + \log(2\cos(\gamma-\omega)) \\ &+ 2\log(2\cos\omega)\,,\end{aligned}$$

and this vanishes because of (15). Further

$$\frac{\partial\phi}{\partial\vartheta} = -\log(2\cos(\vartheta+\omega)) + \log(2\cos(\vartheta-\omega))$$

and this equals $2h(\vartheta)$ because of (16). Herewith the statement (17) is proved.

Notes to Chapter IX

In Section 1 some simple notions of Riemannian geometry are used. Information about these the reader may find in many books on differential geometry. For a short introduction see Lenz [12]. The coordinate systems introduced have been used by many authors.

The expressions for areas and volumes in Sections 2, 3, 4 have been known for a long time. Most of them were already found by Lobatčefskii. For recent studies of the Lobatčefskii function cf. Coxeter [5] and Milnor [15].

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The prefix h used in Chapter III is omitted here.